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AN ARITHMETICAL PROPERTY OF RECURRING SERIES OF THE SECOND ORDER*

BY MORGAN WARD

1. *Statement of Property.* Let us denote by

$$(W_n): \quad W_0, W_1, W_2, \cdots, W_n, \cdots,$$

a sequence of rational integers satisfying the difference equation

$$\Omega_{n+2} = P\Omega_{n+1} - Q\Omega_n, \qquad (P, Q \text{ rational integers}), \tag{1}$$

and let p be an odd prime dividing neither Q nor $P^2-4Q=(\alpha-\beta)^2$, the discriminant of the polynomial

$$x^2 - Px + Q = (x-\alpha)(x-\beta) \tag{2}$$

associated with (1).†

We write as usual $U_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, $V_n=\alpha^n+\beta^n$ for the two Lucas functions built upon the roots α and β of (2).

The distribution of the multiples of p in the corresponding sequences $(U)_n$ and $(V)_n$ is well known: namely, *multiples of p always occur in $(U)_n$*; more specifically, $U_n\equiv 0 \pmod p$ *when and only when* $n\equiv 0 \pmod \tau$, where τ is the restricted period‡ of $(U)_n$ modulo p. In the sequence $(V)_n$, *multiples of p occur when and only when τ is even.* In this case, $V_n\equiv 0 \pmod p$ *when and only when* $n\equiv 0 \pmod{\tau/2}$, $n\not\equiv 0 \pmod\tau$.

For the sequences $(U)_n$ and $(V)_n$ then, we know not only when multiples of p will occur, but where multiples of p will occur. Under the assumption that τ is odd, I propose to obtain a criterion which reduces the problem of determining *when* multiples of p will appear in *any* sequence $(W)_n$ (specified only by its two initial values W_0 and W_1) to the more fundamental (unsolved) problem of determining the characteristic number‡ and restricted period‡ of the Lucas functions associated with any given quadratic polynomial of the form (2).

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* Presented to the Society, June 20, 1934.

† The excluded values of p are evidently trivial for the theorem that follows.

‡ For definitions of these terms, see my *Note on the period of a mark in a finite field*, this Bulletin, vol. 40 (1934), pp. 279–281.

In fact, let P' and Q' be rational integers satisfying the congruences

$$P' \equiv W_0 \pmod{p},$$
$$(4Q - P^2)Q' \equiv W_1^2 - W_0W_1P + W_0^2Q \pmod{p}, \tag{3}$$

and let $U_n' = (\alpha'^n - \beta'^n)/(\alpha' - \beta')$ be the Lucas function associated with the polynomial $x^2 - P'x + Q' = (x - \alpha')(x - \beta')$. Then *a necessary and sufficient condition that the sequence* $(W)_n$ *should contain multiples of* p *is that the restricted period of* U_n' *modulo* p *should be an even divisor of* 2τ.

2. *Illustration.* As a numerical illustration, take $P = 1, Q = -1$, $p = 89$, $W_0 = 1$, $W_1 = 4$. The sequence $(U)_n$ is then the familiar Fibonacci series

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

so that $\tau = 11$. The congruences (3) become $P' \equiv 1 \pmod{89}$, $-5Q' \equiv 11 \pmod{89}$, so that we may take $P' = 1, Q' = -20$. The Lucas sequence $(U')_n$ for $\Omega_{n+2} = \Omega_{n+1} + 20\Omega_n$ runs 0, 1, 1, 21, 41, 461, 1281, 10501, $\cdots$ and by actual computation we find that $U_{22}' \equiv 69 \pmod{89}$. Hence the restricted period of $(U')_n$ modulo 89 is not an even divisor of $2\tau = 22$, and we conclude that all elements of the sequence 1, 4, 5, 9, 14, 23, 37, $\cdots$ are prime to 89, as may be easily verified.

3. *A Preliminary Identity.* My proof of this result is based upon a well known identity in the theory of partitions discovered by Euler,* which we formulate as follows.

Let q be any complex number,

$$[n] \begin{cases} = (q^n - 1)/(q - 1), & q \neq 1, \\ = \lim_{q \to 1} (q^n - 1)/(q - 1) = n, & q = 1. \end{cases}$$

Writing $[n]!$ for $[1][2] \cdots [n]$, $[0]! = 1$, we see that the basic bi-

* *Introductio in Analysin Infinitorum*, 1748, Chapter VII; Netto, *Combinatorik*, 2d ed., 1927, p. 143.

nomial coefficient* $(n;r)$ is defined by $(n;r) = [n]!/\{[n-r]![r]!\}$. This expression, as Gauss showed,† is a polynomial in q which reduces to the ordinary binomial coefficient when $q=1$. The identity in question may now be written as follows:

$$(4) \qquad \sum_{r=0}^{\tau} (\tau; r) q^{r(r+1)/2} z^r = (1 + qz)(1 + q^2z) \cdots (1 + q^\tau z).$$

4. *Proof.* The general term of the sequence $(W)_n$ may be expressed in the form

$$W_n = W_0 U_{n+1} + (W_1 - PW_0) U_n.$$

Thus the restricted period of $(W)_n$ modulo p is a divisor of the restricted period τ modulo p of the Lucas function U_n. Therefore the sequence $(W)_n$ will contain terms divisible by p when and only when the rational integer

$$(5) \qquad \mathfrak{W} = \prod_{n=1}^{\tau} W_n$$

is divisible by p.

Now W_n can also be expressed in the form $W_n = A\alpha^n + B\beta^n$, where the constants A and B are determined by

$$(6) \qquad W_0 = A + B, \qquad W_1 = A\alpha + B\beta.$$

If we let‡ $\beta/\alpha = q$, $B/A = z$, we may write $W_n = A\alpha^n(1+q^n z)$. Therefore by (5) and (4),

$$(7) \qquad \begin{aligned} \mathfrak{W} &= \prod_{n=1}^{\tau} A\alpha^n(1 + q^n z) = A^\tau \alpha^{\tau(\tau+1)/2} \prod_{n=1}^{\tau} (1 + q^n z) \\ &= A^\tau \alpha^{\tau(\tau+1)/2} \sum_{r=0}^{\tau} (\tau; r) q^{r(r+1)/2} z^r. \end{aligned}$$

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* This terminology is due to F. H. Jackson who in recent years has made an extensive study of the basic numbers $[n]$. (See, for example, Proceedings of the London Mathematical Society, (2), vol. 1 (1903–04), pp. 63–68; Proceedings of the Royal Society, (A), vol. 76 (1905), pp. 127–144.)

† *Summatio quarundam Serierum Singularium*, 1808; Works, vol. 2, p. 16.

‡ If A and B are rational integers modulo p, they cannot both be congruent to zero, and we take for A that one which is incongruent to zero modulo p. We have $\alpha \not\equiv 0 \bmod p$, since p was assumed prime to Q.

Now $[n]=(q^n-1)/(q-1)=(\alpha^n-\beta^n)/[(\alpha-\beta)\alpha^{n-1}]=\alpha^{-n+1}U_n$. Hence

$$(n;r) = U_nU_{n-1}\cdots U_{n-r+1}/(U_1U_2\cdots U_r\alpha^{-(n-r)r}).$$

But the first $\tau-1$ of the numbers $U_1, U_2, \cdots, U_\tau$ are prime to p, while U_τ is divisible by p. Hence $(\tau;r)\equiv 0 \pmod p$ unless $r=0$ or $r=\tau$, when it equals one. We therefore obtain from (7) the congruence

$$\mathfrak{W} \equiv A^\tau\alpha^{r(r+1)/2}(1+q^{r(r+1)/2}z^r) \equiv A^\tau\alpha^{r(r+1)/2}+B^r\beta^{r(r+1)/2} \pmod p.$$

Now $\alpha^n=U_n\alpha-QU_{n-1}$, $\beta^n=U_n\beta-QU_{n-1}$. Therefore since τ is odd.*

$$\alpha^{r(r+1)/2} \equiv (U_r\alpha-QU_{r-1})^{(r+1)/2} \equiv (-QU_{r-1})^{(r+1)/2} \pmod p,$$

$$\beta^{r(r+1)/2} \equiv (-QU_{r-1})^{(r+1)/2} \pmod p,$$

and

$$\mathfrak{W} \equiv (-QU_{r-1})^{(r+1)/2}(A^r+B^r) \pmod p. \tag{8}$$

Hence $\mathfrak{W}\equiv 0 \pmod p$ *when and only when* $A^r+B^r\equiv 0 \pmod p$.

Finally, write α', β' for A, B. Then $A^n+B^n=V_n'$, the Lucas function associated with the quadratic polynomial $x^2-P'x+Q'=(x-\alpha')(x-\beta')$.

On referring back to (6) and recalling that p is prime to P^2-4Q, we find that we may assign to P' and Q' the rational integral values specified by the congruences (3). Our theorem now follows immediately from the laws of apparition for multiples of p in the Lucas functions stated in section 1 as applied to the sequence $(V')_n$.

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* It is at precisely this point that the assumption that τ is odd becomes vital. For if we assume that τ is even, we obtain in place of (8) the barren result $\mathfrak{W}=(U_{r+1})^{r/2}(A^{r\alpha^{r/2}}+B^r\beta^{r/2}) \pmod p$.

CONSTRUCTION OF A GROUPLESS AND HEADLESS TRIAD SYSTEM ON 31 ELEMENTS*

BY H. S. WHITE

No doubt the most interesting triad systems are those which admit a large group of substitutions, for example, those of the Heffter headless type,† admitting a cyclic substitution on all the elements. But on a sufficiently large set of elements systems can be constructed which have smaller groups or even no group.‡ The structure of such systems is worth recording for the same reason as, in the theory of algebraic plane curves, not only those with no singular points, but also those with all admissible singularities and actual aggregates of singularities demand enumeration.

The least number of elements in a groupless system is fifteen.§ For thirty-one elements, Reiss's method can be adapted to yield a triad system with no group,‖ but a system built by that method has necessarily a "head", namely a triad (sub-)system on fifteen, a Δ_{15}, which itself has no group. This class of Δ_{31}'s, millions in number, were at their first appearance a surprise, since groupless systems previously known were also headless. But the obvious question was still unanswered: is there any Δ_{31} which is both groupless and headless? I shall here exhibit one such by actual construction, and describe the application of the method of sequences and indices by which it is proved to be both groupless and headless. This method is of course tedious,arousing the wish that some clearer insight and more sweeping form of argument might be attained.

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* Presented to the Society, September 11, 1930.

† L. Heffter, *Ueber Tripelsysteme*, Mathematische Annalen, vol. 49 (1897), p. 101.

‡ H. S. White, *The multitude of triad systems on thirty-one elements*, Transactions of this Society, vol. 16 (1915), pp. 13–19.

§ See reference to DePasquale and Brunel in article by F. N. Cole, *The triad systems of thirteen elemen's* Transactions of this Society, vol. 16 (1913), p. 1.

‖ H. S. White, loc cit

We will take as a starting point a certain Δ_{15} classified by its discoverer, the late Professor F. N. Cole,* as VI, 2, and by Miss L. D. Cummings† as Groupless No. 36. It will be used once with numbers as elements, the integers from 1 to 15; and a second time, according to the Reiss method, with letters from *a* to *r*, omitting *i*, and Reiss's splicing element shall be denoted by *s*. To forestall a group of order 2, I shall derange arbitrarily the columns of the second set, reversing by threes, that is, exchanging, before splicing, columns headed by *sa* and *sc*, also *sd* and *sf*, etc. There results a groupless Δ_{31} having two heads (the two Δ_{15}'s). Arranged under the elements of the first of these Δ_{15}'s, in 15 columns, it is the following array. One misprint has been corrected, Cole's triad 8 11 15 being changed to 9 11 15 as of course it must have been in his MS.

1	2	3	4	5	6	7	8
2 3	1 3	1 2	1 5	1 4	1 7	1 6	1 9
4 5	4 6	4 8	2 6	2 7	2 4	2 5	2 10
6 7	5 7	5 12	3 8	3 12	3 14	3 11	3 4
8 9	8 10	6 14	7 15	6 9	5 9	4 15	5 15
10 11	9 12	7 11	9 10	8 15	8 11	8 13	6 11
12 13	11 14	9 13	11 12	10 14	10 13	9 14	7 13
14 15	13 15	10 15	13 14	11 13	12 15	10 12	12 14
s c	*s b*	*s a*	*s f*	*s e*	*s d*	*s j*	*s h*
a b	*a c*	*b c*	*a g*	*a d*	*a e*	*a h*	*a j*
d h	*d f*	*d e*	*b d*	*b g*	*b f*	*b m*	*b k*
e m	*e g*	*f g*	*c p*	*c m*	*c h*	*c n*	*c d*
f p	*h k*	*h j*	*e j*	*f j*	*g r*	*d k*	*e r*
g l	*j m*	*k l*	*h l*	*h r*	*j k*	*e f*	*f l*
j n	*l p*	*m n*	*k n*	*k p*	*l m*	*g p*	*g n*
k r	*n r*	*p r*	*m r*	*l n*	*n p*	*l r*	*m p*

9	10	11	12	13	14	15	*r*
1 8	1 11	1 10	1 13	1 12	1 15	1 14	1 *k*
2 12	2 8	2 14	2 9	2 15	2 11	2 13	2 *n*
3 13	3 15	3 7	3 5	3 9	3 6	3 10	3 *p*

* F. N. Cole, *Complete classification of the triad systems on fifteen elements*, Memoirs of the National Academy of Sciences, vol. 14 (1919), Second Memoir, Part 4.

† Louise D. Cummings, in the Memoir just cited, Part 5.

4	10	4	9	4	12	4	11	4	14	4	13	4	7	4	*m*
5	6	5	14	5	13	5	15	5	11	5	10	5	8	5	*h*
7	14	6	13	6	8	7	10	6	10	7	9	6	12	6	*g*
11	13	7	12	9	15	3	14	7	8	8	12	9	11	7	*l*
s	*g*	*s*	*m*	*s*	*l*	*s*	*k*	*s*	*r*	*s*	*p*	*s*	*n*	8	*e*
a	*f*	*a*	*n*	*a*	*k*	*a*	*l*	*a*	*p*	*a*	*r*	*a*	*m*	9	*d*
b	*e*	*b*	*j*	*b*	*p*	*b*	*h*	*b*	*n*	*b*	*l*	*b*	*r*	10	*f*
c	*l*	*c*	*e*	*c*	*g*	*c*	*r*	*c*	*k*	*c*	*f*	*c*	*j*	11	*j*
d	*r*	*d*	*l*	*d*	*m*	*d*	*j*	*d*	*g*	*d*	*n*	*d*	*p*	12	*c*
h	*n*	*f*	*r*	*e*	*n*	*e*	*p*	*e*	*h*	*e*	*k*	*e*	*l*	13	*s*
j	*p*	*g*	*k*	*f*	*h*	*f*	*n*	*f*	*m*	*g*	*j*	*f*	*k*	14	*a*
k	*m*	*h*	*p*	*j*	*r*	*g*	*m*	*j*	*l*	*h*	*m*	*g*	*h*	15	*b*

To eradicate the two heads, I choose the column of triads containing 1, and that containing *r*, which latter has been collected from the first fifteen and placed after them in the above table. Between these columns the tetrad relation* occurs three times. I select the triads 1 6 7, 1 *g l*, and *r* 6 *g*, *r* 7 *l*, and replace them by 1 6 *g*, 1 7 *l*, and *r* 6 7 , *r g l*. After this alteration (which requires corresponding alterations in columns under 6 and 7), the upper half and lower half of that 15×15 array no longer form mutually exclusive closed sets, letters in one and integers in the other, for there are now three mixed triads containing one letter each; that is, six mixed pairs in the columns 1, 6, 7. This fact enables me to cut corners in the argument. All triads of the Δ_{31} are listed already in these 15 columns, but for detailed study the 16 additional columns are written out, but not in full. This part need not be given in all detail, since a bare outline will enable anyone to repeat it. I have formed the sequences and listed the indices, complete for columns 1 to 9 inclusive, and enough others so that at least nine indices are listed for triads which contain any particular element. This minimum proves to be sufficiently large for the present purpose.

VERIFICATION. (1) *There is no* Δ_{15} *contained in this* Δ_{31}. If there were, its 15 elements would occur each in 7 triads, and the sequences derived from those triads would constitute a closed system, every pair of elements present occurring twice, in opposite orders, with indices totaling 14 in each column concerned. For

* Memoir cited, p. 73 and p. 70. "Quadrangular" = "Tetrad".

the complete Δ_{31} the indices total 28 for each column. I examine therefore the sequences belonging to each triad separately, noting how many are related by the appearing of one pair in two sequences. Where the index total of related sequences forming a closed set under a triad is 16 or more, neither that set nor the remainder under that triad can belong to a Δ_{15}. Such is the case with at least 9 of the triads in every column; and we need examine no further, since $15-9=6<7$, and it is therefore certain that no element can be found in 7 triads, to form a closed system with the 14 associated elements.

(2) *There is no substitution, aside from the identity, which can transform this* Δ_{31} *into itself.* Any such substitution would replace at least one element by a different one; and the set of indices associated with the former would necessarily be identical with the set belonging to the latter. Now having the *complete* table of indices for nine columns, I compare them among themselves and find no two identical. Then I compare the *partial* set that has been formed for each of the remaining 22 columns with each of the first 9, and find in every partial column at least one index that does not occur in any complete column. Lastly, I examine, for every two of the 22 partial columns, the twice nine indices which they exhibit. If, when completed, the two sets of columns would become identical, the partial columns ought to show at least 4 indices repeated $(18-14=4)$, or the equivalent. But this does not happen in any two partial columns of the 22, and we may omit the labor of carrying these columns to completion. The quality of grouplessness is fully certified.

Let me repeat for the benefit of anyone who may wish to check this proof: in verifying (1) it was frequently necessary to cite the fact that, after the tetrad alteration, neither integers alone nor letters alone are, in this array, a closed set.

It remains an interesting question whether 31 is the least number of elements above 15 for which a groupless and headless triad system can exist.

VASSAR COLLEGE

ON THE ENUMERATION OF MAGIC CUBES*

BY D. N. LEHMER

1. *Introduction.* Assume the cube with one corner at the origin and the three edges at that corner as axes of reference. In a magic cube the sum of the numbers in any line parallel to any one of the three axes is the same, and since the sum of all the numbers up to N^3 is $N^3(N^3+1)/2$, the sum in each of the rows must be $N(N^3+1)/2$. The square array in any plane parallel to a face of the cube forms a magic square according to the more general definition of a magic square as one in which the sum in each row and column is the same even if the numbers involved are other than the numbers from 1 to N^2 inclusive.

2. *Transformations.* We will call the square array in any plane parallel to a face of the cube a *slab*. It is clear that any permutation of a set of parallel slabs among themselves will not affect the magic property of a magic cube. There are manifestly $(N!)^3$ of these permutations, and by means of them we can bring any element into any desired cell of the cube. In particular we can bring the element N^3 to the origin, and after that we can, by a further permutation of the slabs, arrange the elements which lie on the three axes so that they read in descending order of magnitude from the origin out.

Again, by a rotation of the cube through an angle of 120° about the diagonal through the origin, we may make a cyclic permutation of the three axes. We may then assume that the element next to the origin on the x axis is larger than either of those next to the origin on the other two axes. Further, by a reflection of the whole cube in the xy plane followed by a rotation about the x axis of 90°, we may interchange the elements of the y and z axes without disturbing those on the x axis. We may then assume that the element next to the origin on the y axis is larger than that next to the origin on the z axis. We then define a *normal* cube as one which has the element N^3 at the origin with the elements on the three axes arranged in descending order of magnitude from the origin out, and also such

* Presented to the Society, June 20, 1934.

that the elements next to the origin are in descending order of magnitude reading from the x to the y to the z axis in order. There is one and only one normalized cube obtainable from a given cube by the above $2 \cdot 3 \cdot (N!)^3$ transformations. The total number of different magic cubes of order N is therefore equal to the number of normalized cubes of that order multiplied by $6(N!)^3$.

3. *Magic Cubes of Order Three.* There are no cubes of order 2. The magic sum would be 9 and, the largest element being 8, only the element 1 could lie in the same row with it. For the magic cube of order 3 the magic sum is 42 and the other elements on the axes beside the element 27 must add to 15. There are 7 pairs of elements available for the axes: (14, 1) (13, 2) (12, 3) (11, 4) (10, 5) (9, 6) (8, 7). From these we may obtain 35 different types of cubes as follows:

x axis	y axis	z axis
(14, 1)	(13, 2)	(12, 3), (11, 4), (10, 5), (9, 6) (8, 7)
(14, 1)	(12, 3)	(11, 4), (10, 5), (9, 6), (8, 7)
(14, 1)	(11, 4)	(10, 5), (9, 6), (8, 7)
(14, 1)	(10, 5)	(9, 6), (8, 7)
(14, 1)	(9, 6)	(8, 7)
(13, 2)	(12, 3)	(11, 4), (10, 5), (9, 6), (8, 7)
(13, 2)	(11, 4)	(10, 5), (9, 6), (8, 7)
(13, 2)	(10, 5)	(9, 6), (8, 7)
(13, 2)	(9, 6)	(8, 7)
(12, 3)	(11, 4)	(10, 5), (9, 6), (8, 7)
(12, 3)	(10, 5)	(9, 6), (8, 7)
(12, 3)	(9, 6)	(8, 7)
(11, 4)	(10, 5)	(9, 6), (8, 7)
(11, 4)	(9, 6)	(8, 7)
(10, 5)	(9, 6)	(8, 7)

To exhibit the method of examining each of these 35 cases for possible magic cubes we give in detail the examination for the second case where the entries along the three axes are in order: (14, 1), (13, 2) and (11, 4). We first list the possible slabs in the xy plane which have for their top line the entries 27, 14, 1

and for the left-hand column the entries 27, 13, 2. These are got by assigning successive available elements for the center element, the others being then determined. None of these center elements must occur on the three axes, and none must give equal elements in the slab. The first condition bars out the numbers 1, 2, 4, 11, 13, 14 and 27. Besides this the number 8 can not be used as that would give two bottom elements equal to 20. We get thus the following for the xy plane:

27 14 1	27 14 1	27 14 1	27 14 1
13 3 26	13 5 24	13 6 23	13 7 22
2 25 15	2 23 17	2 22 18	2 21 19
27 14 1	27 14 1	27 14 1	27 14 1
13 9 20	13 10 19	13 11 18	13 12 17
2 19 21	2 18 22	2 17 23	2 16 24

Similarly we list the possible slabs in the yz plane. They are:

27 13 2	27 13 2	27 13 2	27 13 2	27 13 2
11 6 25	11 7 24	11 8 23	11 9 22	11 12 19
4 23 15	4 22 16	4 21 17	4 20 18	4 17 21

For the zx plane we also have the following:

27 14 1	27 14 1	27 14 1	27 14 1
11 5 26	11 6 25	11 7 24	11 8 23
4 23 15	4 22 16	4 21 17	4 20 18
27 14 1	27 14 1	27 14 1	
11 10 21	11 12 19	11 16 15	
4 18 20	4 16 22	4 12 26	

We select now a slab from the xy plane, and a slab from the yz plane and a slab from the zx plane, taking care that none of the entries (except, of course, those that may lie on the axes) be repeated in any two slabs. Starting with the xy slabs in order, the first set that appears is

27 14 1	27 13 2	27 14 1
13 3 26	11 7 24	11 8 23
2 25 15	4 22 16	4 20 18

Given these three slabs and the center of the cube, the rest of the elements of the cube are determined. This center element

must again be chosen from the elements not already found in the three slabs. Since it lies in line with the three central elements of the slabs it is further restricted since the sum of the elements in those lines must be 42. Now the only pairs which with 3 will give 42 are (26, 13), (25, 14), (24, 15), (23, 16), (22, 17), (21, 18) and (20, 19), and neither number in any pair must have been used before. But this condition rules out every one of the pairs, and so there can be no cube with the above set of slabs in the three planes of reference.

The other sets of three slabs are treated similarly and all are ruled out at once except the set:

27 14 1	27 13 2	27 14 1
13 3 26	11 9 22	11 7 24
2 25 15	4 20 18	4 21 17

Here the central number 3 rules out all the available pairs except (23, 16), neither of which numbers appear in any of the slabs. When we try these numbers on the central element 7, they are still both usable, since 7 and 23 demand 12 and 7 and 16 demand 19 and both of these numbers are still available. But the central element 9 of the third slab rules out the entry 16, since 9 and 16 demand 17 which is in the third slab already. But the number 23 is still left since 9 and 23 determine 10 which is still unused.

We proceed to derive then all the other elements of the cube, given the above three slabs and the central element 23. The result is our first magic cube. We give the three horizontal slabs in order from the bottom up:

	27 14 1	11 7 24	4 21 17
(1)	13 3 26	9 23 10	20 16 6
	2 25 15	22 12 8	18 5 19

In the same way all of the other cases were examined. Naturally certain short cuts appeared to abridge the work. The examinations were made three times, and in the third the cubes were normalized by putting the element 1 in the corner of the cube and arranging the elements on the three axes in ascending order out from the origin. Three and only three other cubes were obtained. The others are:

	27 14 1	10 9 23	5 19 18
(2)	13 3 26	8 22 12	21 17 4
	2 25 15	24 11 7	16 6 20
	27 14 1	10 9 23	5 19 18
(3)	11 7 24	6 20 16	25 15 2
	4 21 17	26 13 3	12 8 22
	27 13 2	10 8 24	5 21 16
(4)	11 9 22	6 19 17	25 14 3
	4 20 18	26 15 1	12 7 23

Each of these normalized cubes represents a group of 1296, so that there are altogether 5,184 magic cubes of order 3.

THE UNIVERSITY OF CALIFORNIA

EXTENSION OF RANGE OF FUNCTIONS*

BY E. J. McSHANE

A well known and important theorem of analysis states that a function $f(x)$ which is continuous on a bounded closed set E can be extended to the entire space, preserving its continuity. Let us consider a metric space S and a function $f(x)$ defined and possessing a property P on a subset E of S. We shall for the sake of brevity say that $f(x)$ *can be extended to S preserving property P*, if there exists a function $\phi(x)$, defined and possessing property P on all of S, which is equal to $f(x)$ for all x on E. Our present object is to establish an easily proved theorem which both includes the classical theorem stated above, and also shows that functions satisfying a Lipschitz or Hölder condition on an arbitrary set E can be extended to S preserving the Lipschitz or Hölder condition. An advantage of the present procedure is that it yields an explicit formula for the extension.†

* Presented to the Society, June 20, 1934.

† After this paper was submitted for publication, the author found that Hassler Whitney had already indicated a simple proof that a function continuous on a bounded closed set can be extended to be continuous on all space, the method of extension being almost identical with the present one. (H. Whitney, Transactions of this Society, vol. 36 (1934), footnote on p. 63).

In order to exhibit the simplicity of the method we first consider the special case of Lipschitzian functions. In our metric space S we denote the distance of points x_1, x_2 by $\|x_1, x_2\|$.

THEOREM 1. *Let the real-valued function $f(x)$ be defined on a subset E of the metric space S and satisfy the Lipschitz condition*

$$|f(x_1) - f(x_2)| \leqq M\|x_1, x_2\| \tag{1}$$

on E. Then $f(x)$ can be extended to S preserving the same Lipschitz condition.

For every x of S we define $\phi(x)$ to be the least upper bound of $f(\bar{x}) - M\|\bar{x}, x\|$ for all $\bar{x}$ of E: $\phi(x) = \overline{B}(f(\bar{x}) - M\|\bar{x}, x\|)$. If x is in E, we have $\phi(x) = f(x)$; for by (1), $f(\bar{x}) - M\|\bar{x}, x\| \leqq f(x)$, and this upper bound $f(x)$ is attained for $\bar{x} = x$. Also, $\phi(x)$ satisfies the Lipschitz condition. For let x, x' be any two points of S, and suppose to be specific that $\phi(x') \geqq \phi(x)$. Then, unless $\phi(x) = \phi(x') = \infty$,

$$\begin{aligned} 0 \leqq \phi(x') - \phi(x) &= \overline{B}(f(\bar{x}) - M\|\bar{x}, x'\|) - \overline{B}(f(\bar{x}) - M\|\bar{x}, x\|) \\ &\leqq \overline{B}[(f(\bar{x}) - M\|\bar{x}, x'\|) - (f(\bar{x}) - M\|\bar{x}, x\|)] \\ &= \overline{B}M(\|\bar{x}, x\| - \|\bar{x}, x'\|) \leqq M\|\bar{x}, x'\|. \end{aligned}$$

Therefore, unless $\phi(x) = \phi(x') = \infty$, we have

$$|\phi(x) - \phi(x')| \leqq M\|x, x'\|. \tag{2}$$

In particular, if x is arbitrary and x' is in E, $\phi(x')$ is finite; then (2) holds. Hence, by (2), $\phi(x)$ is always finite and (2) always holds.

Before proceeding to the general case we prove a lemma concerning moduli of continuity. If $f(x)$ be defined on E, we define its *least modulus of continuity* $\omega_0(t)$ by the relation

$$\omega_0(t) = \overline{B}|f(x) - f(\bar{x})| \text{ for all } x, \bar{x} \text{ of } E \text{ such that } \|x, \bar{x}\| \leqq t.$$

We say that $\omega(t)$ *is a modulus of continuity of the function* $f(x)$, if $\omega(t) \geqq \omega_0(t)$; that is, if the inequality $|f(x) - f(\bar{x})| \leqq \omega(t)$ holds for all points x, $\bar{x}$ of E such that $\|x, \bar{x}\| \leqq t$. Clearly $\omega_0(t)$ is non-negative and monotonic increasing; and by definition, $f(x)$ is uniformly continuous on E if $\omega_0(t)$ tends to zero with t.

LEMMA. *If $\omega_0(t)$ is defined and non-negative for $t \geqq 0$, and there exist constants h, k such that $\omega_0(t) \leqq ht + k$, then there exists a func-*

tion $\omega(t) \geqq \omega_0(t)$ which is continuous and concave for all $t>0$. Moreover, if ω_0 tends to 0 with t, we may require $\omega(t)$ to do the same.*

In the (t, u) plane we consider the totality of all lines $u=at+b$ for which the inequality $\omega_0(t) \leqq at+b$ holds for all t. For each such line the region $t \geqq 0$, $u \leqq at+b$ is a convex region; hence the common part II of all these regions is convex. Its upper boundary is a concave curve $u=\omega(t)$; and since the curve $u=\omega_0(t)$ lies in II, we have $\omega(t) \geqq \omega_0(t)$. On any interval $0 \leqq t \leqq t_0$, the function $\omega(t)$ is bounded; being concave, it is therefore continuous. Suppose further that $\omega_0(t) \to 0$; then to every $\epsilon>0$ there corresponds a δ such that $0 \leqq \omega_0(t) \leqq \epsilon$ for $t \leqq \delta$. We can easily find a linear function $u=at+\epsilon$, $a>0$, such that $at+\epsilon > ht+k \geqq \omega_0(t)$ for $t>\delta$. Then $\omega(t) < at+\epsilon$, so that $0 \leqq \lim_{t \to 0} \omega(t) \leqq \epsilon$. This being true for every $\epsilon>0$, the limit of $\omega(t)$ must be zero.

As examples of functions $f(x)$ for which $\omega_0(t)$ satisfies the hypotheses of the lemma, we have the following.

(a) All bounded uniformly continuous functions. For if $|f| \leqq M$, then $\omega_0(t) \leqq 2M$.

(b) All functions satisfying a Hölder condition

$$|f(x_1) - f(x_2)| \leqq M \|x_1, x_2\|^{\alpha}, \qquad (0 < \alpha \leqq 1).$$

For then $\omega_0(t) \leqq Mt^{\alpha} \leqq M(1+t)$.

THEOREM 2. *If $f(x)$ is a real function defined on a subset E of a metric space S, and $f(x)$ has a modulus of continuity $\omega(t)$ which is concave for $t \geqq 0$ and which approaches zero with t, then $f(x)$ can be extended to S preserving the modulus of continuity $\omega(t)$.*

As a first remark, we observe that $\omega(t)$ is monotonic increasing; otherwise it would be negative for sufficiently large t, which is absurd. We now define

$$\phi(x) \equiv \overline{B}(f(\bar{x}) - \omega(\|\bar{x}, x\|)),$$

where x ranges over E. For all x of E we have $\phi(x)=f(x)$, because by hypothesis $f(\bar{x}) - \omega(\|\bar{x}, x\|) \leqq f(x)$, and this upper bound $f(x)$ is attained for $\bar{x}=x$.

Let now x and x' be any two points of S for which $\phi(x)$ and $\phi(x')$ are not both infinite, and suppose to be specific that

* Concave downwards.

$\phi(x) \geqq \phi(x')$. Then, remembering the monotoneity and concavity of $\omega(t)$, we see that

$$\begin{aligned} 0 \leqq \phi(x) - \phi(x') &= \overline{B}(f(\bar{x}) - \omega(\|\bar{x}, x\|)) - \overline{B}(f(\bar{x}) - \omega(\|\bar{x}, x'\|)) \\ &\leqq \overline{B}[(f(\bar{x}) - \omega(\|\bar{x}, x\|)) - (f(\bar{x}) - \omega(\|\bar{x}, x'\|))] \\ &\leqq \overline{B}[\omega(\|\bar{x}, x\| + \|x, x'\|) - \omega(\|\bar{x}, x\|)] \\ &\leqq \omega(\|x, x'\|). \end{aligned}$$

Hence, unless $\phi(x) = \phi(x') = \infty$, the inequality

$$|\phi(x) - \phi(x')| \leqq \omega(\|x, x'\|) \tag{3}$$

holds. If we choose x' in E, $\phi(x')$ is finite, so that (3) holds for all x. This implies that $\phi(x)$ is always finite, so that (3) holds for all x and x' of S without exception.

COROLLARY 1. *If $f(x)$ satisfies on E a Lipschitz or Hölder condition*

$$|f(x_1) - f(x_2)| \leqq M\|x_1, x_2\|^{\alpha}, \qquad (0 < \alpha \leqq 1),$$

then $f(x)$ can be extended to S preserving the Lipschitz or Hölder condition.

For in Theorem 2 we can take $\omega(t) = Mt^{\alpha}$.

COROLLARY 2. *If $f(x)$ is bounded and uniformly continuous on E, it can be extended to S preserving the uniform continuity and the bounds.*

For by the lemma, $f(x)$ has a modulus of continuity $\omega(t)$ satisfying the hypotheses of Theorem 2. We can therefore find a function $\overline{\phi}(x)$, defined and with modulus of continuity $\omega(t)$ on S and coinciding with $f(x)$ on E. If m, M are, respectively, the lower and upper bounds of $f(x)$ on E, we define $\phi(x) = \overline{\phi}(x)$ where $m \leqq \overline{\phi}(x) \leqq M$, $\phi(x) = M$ where $\overline{\phi}(x) > M$, $\phi(x) = m$ where $\overline{\phi}(x) < m$. Then $\phi(x)$ has bounds m and M, and $\omega(t)$ serves as a modulus of continuity for $\phi(x)$; and on E we have $\phi = \overline{\phi} = f$.

As a special case of Corollary 2, a function $f(x)$ defined and continuous on a bounded closed subset E of n-dimensional euclidean space S_n can be extended to S_n, preserving continuity and the bounds. For $f(x)$ is necessarily bounded and uniformly continuous on E.

COROLLARY 3. *If $f(x)$ is uniformly continuous on E, it can be extended to be continuous on S.*

If x' is a limit point of E, $f(x)$ has a unique limit as x approaches x', because of the uniform continuity of f. Hence we can extend f at once to the closure $\overline{E}$ of the set E, preserving uniform continuity. If we define $p(x)$ to be the distance of x from the set $\overline{E}$, then $p(x)=0$ for x in $\overline{E}$ and $p(x)>0$ for all other x. Let us now define $f_1(x)=(2/\pi)$ arc tan $f(x)$; then $|f_1|<1$ for all x in $\overline{E}$. Since $|f_1(x)-f_1(x')|<|f(x)-f(x')|$, it is clear that f_1 is uniformly continuous on $\overline{E}$. By Corollary 2 there exists a function $\bar{\phi}(x)$, continuous on S, coinciding with f_1 on $\overline{E}$, and such that $|\bar{\phi}|\leqq 1$. If we define $\phi(x)=\bar{\phi}(x)(1+p(x))^{-1}$, then for all x in $\overline{E}$ we have $|\phi(x)|=|\bar{\phi}(x)|<1$, while for all other x we have $|\phi(x)|<|\bar{\phi}(x)|\leqq 1$. Hence the inequality $|\phi(x)|<1$ holds for all x. The function $\psi(x)=\tan(\pi\phi(x)/2)$ is then the desired extension of f. For on $\overline{E}$ we have $\phi(x)=\bar{\phi}(x)=f_1(x)$, so that $\psi(x)=f(x)$; and since ϕ is continuous and less than 1 in absolute value, $\psi(x)$ is continuous for every x.

If we compare Corollary 3 with Theorem 2 (and the lemma), we see that it has weaker hypotheses, since we do not assume that $\omega_0(t)$ is less than a linear function $ht+k$, and it also has a weaker conclusion, since the extension of $f(x)$ is not necessarily uniformly continuous. As a matter of fact, if S is a linear space it can be shown with little difficulty that in order for $f(x)$ to be extensible to a uniformly continuous function S it is necessary that $\omega_0(t)$ be less than some linear function.

Before proceeding to Corollary 4 we introduce some new notation. We denote by $K(x, r)$ the sphere with center x and radius r; that is, the set of all x' such that $\|x', x\|<r$. The surface of the sphere we denote by $K^*(x, r)$; this is the set of all x for which $\|x', x\|=r$. The sum of $K(x, r)$ and $K^*(x, r)$ is $\overline{K}(x, r)$. These are all metric spaces, if not vacuous.

COROLLARY 4. *If $f(x)$ is bounded and uniformly continuous on every sphere $K(x, r)$ of S, it can be extended to be continuous on S, the extension being bounded and uniformly continuous on every sphere $K(x, r)$.*

As in Corollary 3, we may assume that the range of definition E is closed. We choose a point x_0 of S, and denote by K_n the sphere $K(x_0, n)$. For each n we can extend $f(x)$ (considered as

a function on $E \cdot K_{n+1}$ alone) to be bounded and uniformly continuous on $E \cdot K_{n+1} + K_n^*$, by Corollary 2. For each n, this extended f is bounded and uniformly continuous on the set $E \cdot (\overline{K}_n - K_{n-1}) + K_{n-1}^* + K_n^*$, and hence can be extended to be bounded and uniformly continuous on $\overline{K}_n - \overline{K}_{n-1}$. The extensions of f thus obtained piece together to form a function $\phi(x)$ which is bounded and uniformly continuous on each K_n and continuous on all of S. Every sphere $K(x, r)$ lies in some K_n, hence $\phi(x)$ is bounded and uniformly continuous on $K(x, r)$.

As a consequence of Corollary 4, *every function defined and continuous on a closed set E of euclidean n-space S_n can be extended to be continuous on S_n.*

Corollary 4 does not include Corollary 3. Suppose for example that S is a Hilbert space and $x_1, x_2, \cdots$ is a normed orthogonal set. If we assign $f(x)$ any values on the x_i, it can be extended (by Corollary 3) to be continuous on S, because $\omega_0(t) = 0$ for $0 \leqq t < 2^{1/2}$. But the hypotheses of Corollary 4 are not satisfied unless the values $f(x_i)$ are bounded.

A function $f(x, y)$, defined on a set E in the (x, y) plane and absolutely continuous in the sense of Tonelli, can not always be extended even to be of limited total variation, not even if E be an open set plus its boundary. For let I_n denote the interval $2^{-2n-1} \leqq x \leqq 2^{-2n}$, $(n = 0, 1, \cdots)$, and let E be the set (x in $\sum I_n$, $0 \leqq y \leqq 1$) plus the intervals $x = 0$, $0 \leqq y \leqq 1$. If x is in I_n, we define $f(x, y) = (-1)^n (n+1)^{-1}$; for $x = 0$, we set $f(x, y) = 0$. This function is continuous on E and absolutely continuous in the sense of Tonelli in the interior of E. But any function $\phi(x, y)$ defined on the unit square Q: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and coinciding with f on E would fail to have limited total variation, for on any abscissa through Q the function would pass through the values $1, -1/2, +1/3, -1/4, \cdots$. It does not help to require E to be simply connected. For to the set E above let us add the part of Q below the line $y = x$, and in each of the trapezoids thus added let us define $f(x, y)$ to be linear in x and joining continuously to the values already assigned on E. The new set E_1 consists of a simply connected open set plus its boundary; $f(x, y)$ is absolutely continuous, and still any extension to Q must fail to have limited total variation.

PRINCETON UNIVERSITY

THE PRINCIPAL MATRICES OF A RIEMANN MATRIX*

BY A. A. ALBERT

1. *Introduction.* A matrix ω with p rows and $2p$ columns of complex elements is called a *Riemann matrix* if there exists a rational $2p$-rowed skew-symmetric matrix C such that

$$\omega C\omega' = 0, \qquad \pi = i\omega C\bar{\omega}' \tag{1}$$

is positive definite. The matrix C is called *a principal matrix* of ω and it is important in algebraic geometry to know *what are all principal matrices of ω in terms of a given one.* In the present note I shall solve this problem.

2. *Principal Matrices.* A rational $2p$-rowed square matrix A is called a projectivity of ω if

$$\alpha\omega = \omega A \tag{2}$$

for a p-rowed complex matrix α. The Riemann matrices ω have recently† been completely classified in terms of their projectivities; so we may regard all the projectivities A of ω as known.

A projectivity A is called symmetric if $CA'C^{-1}=A$. Let A be a symmetric projectivity so that if $B=AC$, then $B'=(AC)'=-CA'=-AC=-B$ is a skew-symmetric matrix. Then iAC is Hermitian and so must be

$$\delta = \omega(iAC)\bar{\omega}' = \alpha(i\omega C\bar{\omega}') = \alpha\pi. \tag{3}$$

Now π is positive definite so that $\pi=\rho\bar{\rho}'$, where ρ is non-singular. Then $\pi^{-1}=(\bar{\rho}')^{-1}\rho^{-1}=\bar{\sigma}'\sigma$ with σ non-singular. Hence $\alpha=\delta\pi^{-1}=\delta\bar{\sigma}'\sigma$ and

$$\sigma\alpha\sigma^{-1} = \sigma\delta\bar{\sigma}'. \tag{4}$$

The matrix $\sigma\delta\bar{\sigma}'$ is evidently Hermitian and it is well known that then $\sigma\delta\bar{\sigma}'$ and the similar matrix α have *only simple ele-*

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* Presented to the Society, September 7, 1934.

† See my paper *A solution of the principal problem in the theory of Riemann matrices*, Annals of Mathematics, October, 1934.

mentary divisors and *all real characteristic roots.* Thus $\alpha=\beta\gamma\beta^{-1}$, where γ is a real diagonal matrix.

Write

$$\Omega=\begin{pmatrix}\omega\\ \bar{\omega}\end{pmatrix},$$

so that, as is well known, and may easily be computed,

$$A=\Omega^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \bar{\alpha}\end{pmatrix}\Omega=\Lambda\Gamma\Lambda^{-1}, \tag{5}$$

where

$$\Gamma=\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix},\qquad \Lambda=\Omega^{-1}\begin{pmatrix}\beta & 0\\ 0 & \bar{\beta}\end{pmatrix}. \tag{6}$$

Then A is similar to the real diagonal matrix Γ and we have proved the following theorem.*

THEOREM 1. *A symmetric projectivity of a Riemann matrix has all simple elementary divisors and all real characteristic roots.*

We may now determine all principal matrices of a given Riemann matrix ω with a given principal matrix C. Let B be a second principal matrix of ω so that $\omega B\omega'=0$. It is well known that $BC=A$ is a projectivity of ω. In fact $\alpha\omega=\omega A$, where $\alpha=\delta\pi^{-1}$ is defined by (3). Moreover $B'=-B$, so that

$$(AC)'=C'A'=-CA'=-AC,$$

and $CA'C^{-1}=A$. Hence $A=BC^{-1}$ is a symmetric projectivity of ω.

The matrix $\delta=i\omega B\bar{\omega}$ is positive definite if B is a principal matrix of ω. Hence $\sigma\delta'\bar{\sigma}'$ is positive definite and has all positive characteristic roots. The matrices α and γ defined above are similar to $\sigma\alpha\sigma^{-1}=\sigma\delta\bar{\sigma}'$ and have the same characteristic roots, so that the diagonal matrix Γ, whose diagonal elements are these characteristic roots repeated, has all positive diagonal elements. Then A, which is similar to Γ, has all positive characteristic roots.

Conversely, let A be a symmetric projectivity of ω with all positive characteristic roots. Then Γ has all positive diagonal

* The proof by the use of (4) was suggested by certain analogous considerations of N. Jacobson.

elements, α has all positive characteristic roots and so has $\sigma\alpha\sigma^{-1}=\sigma\delta\bar{\sigma}'$. But $\sigma\delta\bar{\sigma}'$ is an Hermitian matrix with characteristic roots all positive. Then $\sigma\delta\bar{\sigma}'$ is positive definite and so is $\delta=i\omega A C\omega'$. Moreover, if $B=AC$, then

$$\omega B\omega'=\omega A C\omega'=\alpha\omega C\omega'=0$$

and B is a principal matrix of ω. We have proved the following result.

THEOREM 2. *Let ω be a Riemann matrix with principal matrix C and let A range over the set of all symmetric projectivities of ω which have positive characteristic roots. Then a rational matrix B is a principal matrix of ω if and only if $B=AC$ with A in the above set.*

3. *Pure Riemann Matrices of the First Kind.* The problem of determining what projectivities of ω are symmetric with all characteristic roots positive is, in general, a complicated one. We may nevertheless solve this problem for the case where ω is a pure Riemann matrix of the first kind.

The multiplication algebra of a pure Riemann matrix is a division algebra D. The centrum of D is a field represented by a field $R(S)$ of all polynomials with rational coefficients of a projectivity S of ω. Algebra D is of the first or second kind according as S is or is not symmetric.

If D is of the first kind, then I have proved* that every projectivity of ω has the form $p(S)$ in $R(S)$ or the form

$$\alpha_1+\alpha_2X+\alpha_3Y+\alpha_4XY, \tag{7}$$

with $\alpha_1, \cdots, \alpha_4$ in $R(S)$, such that

$$YX=-XY, \quad X^2=\xi, \quad Y^2=\eta, \quad (\xi, \eta \text{ in } R(S)). \tag{8}$$

The order of the set of all symmetric projectivities of ω is its singularity index k. If S is symmetric and $R(S)$ has order t, then $k=t$ or $k=3t$ according as we may not or may take both X and Y symmetric, while $k=t$ if D is equivalent to $R(S)$.

Let first $k=t$ so that every symmetric projectivity of ω is in $R(S)$, and let the characteristic roots of S be $\sigma_1, \cdots, \sigma_t$. Then

* Annals of Mathematics, vol. 33 (1932), pp. 311–318.

if $A = p(S)$, the characteristic roots of A are $p(\sigma_j)$ and we have the following theorem.

THEOREM 3. *Let ω be a pure Riemann matrix of the first kind with projectivity algebra D_0 over $R(S)$ having singularity index $k = t$. Then the principal matrices of ω are the matrices*

$$p(S)C, \tag{9}$$

where $p(S)$ is a polynomial in S with rational coefficients such that

$$p(\sigma_j) > 0, \qquad (j = 1, \cdots, t). \tag{10}$$

Next let $k = 3t$ so that every symmetric projectivity of ω has the form

$$A = p_1(S) + p_2(S)X + p_3(S)Y. \tag{11}$$

Then A satisfies the equation in an indeterminate α

$$[\alpha - p_1(S)]^2 = [p_2(S)]^2\xi + [p_3(S)]^2\eta. \tag{12}$$

Hence the characteristic roots of A are the numbers

$$p_1(\sigma_j) \pm \{[p_2(\sigma_j)]^2\xi(\sigma_j) + [p_3(\sigma_j)]^2\eta(\sigma_j)\}^{1/2}.$$

Since X and Y are symmetric we have the well known trivial result

$$\xi(\sigma_j) > 0, \qquad \eta(\sigma_j) > 0. \tag{13}$$

But then the characteristic roots of A are all positive if and only if

$$p_1(\sigma_j) > \{[p_2(\sigma_j)]^2\xi(\sigma_j) + [p_3(\sigma_j)]^2\eta(\sigma_j)\}^{1/2}. \tag{14}$$

We have proved the following theorem.

THEOREM 4. *Let ω be pure with singularity index $k = 3t$ and let $p_1(S)$, $p_2(S)$, $p_3(S)$ be polynomials in S with rational coefficients. Then every principal matrix of ω is given by the set of matrices*

$$[p_1(S) + p_2(S)X + p_3(S)Y]C, \tag{15}$$

with p_1, p_2, p_3 chosen so that (14) *holds.*

THE UNIVERSITY OF CHICAGO

TRANSFINITE SUBGROUP SERIES*

BY GARRETT BIRKHOFF

1. *Summary.* This note contains a proof that the Theorem of Jordan-Hölder can be extended to the case of any series of normal subgroups or, more generally, to the case of what we shall call "T-invariant" subgroups well-ordered in the direction of increasing subgroups. An example is given showing that the replacement of "increasing" by "decreasing" in the preceding sentence renders the proposition false.

Finally, the situation as regards the subgroup-series of compact topological groups homeomorphic with subsets of Cartesian n-space is clarified by two superficial observations.

2. *Definitions and Notation.* Let G be any group; H and K any two subgroups of G. We shall write $H \cap K$ for the *meet* or cross-cut of H and K, and $H \cup K$ for the subgroup generated by the *join* of H and K. The statements $H < K$ and $K > H$ mean that H is contained in, but is different from, K; $H \not< K$ and $K \not> H$ mean that $H < K$ is false. The statement $H \supset K$ means H includes K.

Now let A be the group of all automorphisms, and A_I the subgroup of the inner automorphisms of G, and let T be any subgroup of A containing A_I. The subgroup H will be called *T-invariant* if and only if it is carried into itself under every automorphism of T. It is certain that any T-invariant subgroup is normal.

By a *T-series* of G we shall mean† any set Σ of T-invariant subgroups T_i of G with the two properties:

(i) If $i \neq j$, then either $T_i < T_j$ or $T_i > T_j$.

(ii) To every T-invariant subgroup X of G corresponds a $T_i \in \Sigma$ such that $T_i \not< X$ and $T_i \not> X$.

By a *well-ordered ascending* (*well-ordered descending*) T-series of G is meant one in which every subset has a least (greatest) term.

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* Presented to the Society, September 7, 1934.

† The cases $T = A_I$ and $T = A$ yield under these definitions normal subgroups and chief series, and characteristic subgroups and characteristic series. The cases $A_I < T < A$ yield generalizations.

3. *Extension of the Jordan-Hölder Theorem.* We shall want to use the following rather simple lemma.

LEMMA. *If G is any group, while S, T, and U are T-invariant subgroups of G such that S is a largest T-invariant subgroup of G in T, then either $S \cup U = T \cup U$ or $S \cup U$ is a largest T-invariant subgroup of G in $T \cup U$.*

Suppose the contrary, that G contained a T-invariant subgroup W satisfying $S \cup U < W < T \cup U$. Consider $W \cap T$; evidently $S \subset W \cap T \subset T$, whence $W \cap T = S$ or $W \cap T = T$. But if $W \cap T = T$, then $W \supset T$ as well as $W \supset U$, so that $W \supset T \cup U$ contrary to hypothesis. While if $W \cap T = S$, then since $U \subset W$, $S \cup U = (W \cap T) \cup U = W \cap (U \cup T) = W$, again contrary to hypothesis.*

We are now in a position to prove the following theorem.

THEOREM 1. *Let Σ: $1 = T_1 < T_2 < T_3 < \cdots < T_m = G$ and Σ': $1 = T_1' < T_2' < T_3' < \cdots < T_n' = G$ be any two well-ordered ascending T-series† of G. Then we can establish a $(1,1)$ correspondence between the T_{i+1}/T_i and the T'_{j+1}/T'_j, such that corresponding factor-groups are isomorphic under an isomorphism preserved under every automorphism performed by T.*

For let $T'_{j(i)}$ be the first term of Σ' satisfying $T'_{j(i)} \cup T_i = T'_{j(i)} \cup T_{i+1}$. Then $j(i)$ is evidently single-valued and defined for all i.

Now $j(i)$ is not a limit-number. For since to contain one element of $T_{i+1} - T_i$ and all of T_i would be‡ to contain all of T_{i+1}, we can be sure that§

$$T'_j \cup T_i = \lim_{k \to j} T'_k \cup T_i = \sum_{k<j} T'_k \cup T_i$$

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* We are using the fundamental combinatorial formula for normal subgroups, that if $A \subset C$, then $A \cup (B \cap C) = (A \cup B) \cap C$. See Theorem 26.1 of my paper *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464.

† Here m and n are of course finite or transfinite ordinals.

‡ Since every $(T_k \cup T_i) \cap T_{i+1} = T(i, k)$ is a T-invariant subgroup of G, we must exclude the hypothesis $T_i < T(i, k) < T_{i+1}$.

§ If $S_1 \subset S_2 \subset S_3 \subset \cdots$, then $S_1 \cup S_2 \cup S_3 \cup \cdots = S_1 + S_2 + S_3 + \cdots$ for subalgebras of any algebra whose operations act only on finite sets of elements; for example, groups, rings, and lattices.

contains no elements of $T_{i+1}-T_i$. That is, $j(i)-1$ surely exists.

But $T_{j(i)-1} \cup T_i < T'_{j(i)-1} \cup T_{i+1}$ is by the Lemma a largest T-invariant subgroup of G in $T'_{j(i)} \cup T_i = T'_{j(i)} \cup T_{i+1}$. And we know that $T'_{j(i)-1} \cup T_{i+1} \subset T'_{j(i)} \cup T_{i+1}$; therefore $T_{j(i)-1} \cup T_{i+1} = T'_{j(i)} \cup T_{i+1}$. That is, T_{i+1} is reciprocally the first term T_k of Σ such that $T_k \cup T'_{(i)-1} = T_k \cup T'_{(i)}$.

This establishes a (1, 1) correspondence between the T_{i+1}/T_i and the T'_{j+1}/T'_j. Since under this correspondence the association of each coset of T_{i+1}/T_i or T'_{j+1}/T'_j with that coset of $T_{i+1} \cup T'_{j+1}/T_i \cup T'_j$ containing it defines an isomorphism preserved under every automorphism of T, we have proved Theorem 1.

4. *Simple Counter-Examples.* Let C be the enumerable cyclic group generated by a single element g. Let A_i and B_i denote the (normal) subgroups of G generated by g^{2^i} and g^{3^i}, respectively. It is entirely evident that

$$G > A_1 > A_2 > A_3 > \cdots ; 1 \qquad \text{and}$$

$$G > B_1 > B_2 > B_3 > \cdots ; 1$$

are chief (and composition series in the natural sense of the word.* Yet the first contains only factor-groups of order two, and the second only those of order three. There results the following theorem.

THEOREM 2. *The enumerable cyclic Abelian group has well-ordered descending chief series which do not satisfy the theorem of Jordan-Hölder.*

We must not assume that because one T-series of a group is well-ordered in the direction of increasing subgroups, all of its T-series are. Take the enumerable Abelian group G generated by elements $a_1, a_2, a_3, \cdots$ of order two. Let S_i denote the (normal) subgroup generated by $a_1, \cdots, a_i$, and T_i the (normal) subgroup generated by $a_{i+1}, a_{i+2}, a_{i+3}, \cdots$. Then the S_i and the T_i (with I and G thrown in) constitute a counter-example.

* See, for instance, O. Schreier, *Über den Jordan-Hölderschen Satz*, Hamburg Abhandlungen, vol. 6, pp. 300–302. Since the present paper was written, Schreier's proof has been improved by H. Zassenhaus, *Zum Satz von Jordan-Holder-Schreier*, Hamburger Abhandlungen, vol. 10 (1934), pp. 106–109.

5. *Compact Topological Groups.* The following theorem is loosely related to Theorem 1.

THEOREM 3. *Let G be any compact topological group whose manifold is homeomorphic with a subset of Cartesian n-space. Then any series of closed subgroups of G can be well-ordered in the direction of increasing subgroups.*

For the different group nuclei* are at most $(n+1)$ in number. And the index of the subgroup generated by any one of these nuclei in any larger closed subgroup having the same nucleus is finite.

But if we restrict ourselves to *closed* T-invariant subgroups, then the proof of Theorem 1 breaks down. For consider the additive group of residues modulo unity. The subgroups generated by 1/2, 1/4, 1/8, $\cdots$ form one chief series, and those generated by 1/3, 1/9, 1/27, $\cdots$ a second one, and yet the two have not a single factor-group in common.

SOCIETY OF FELLOWS, HARVARD UNIVERSITY

LOCI OF m-SPACES JOINING CORRESPONDING POINTS OF $m+1$ PROJECTIVELY RELATED n-SPACES IN r-SPACE†

BY B. C. WONG

Let $m+1$ n-spaces $S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m+1)}$ be given in general positions in an r-space S_r. It is convenient, but not necessary, to let $r=mn+m+n$. We shall assume that the given n-spaces are in an S_{mn+m+n}. Now suppose that these n-spaces are all projectively related, that is, to a given subspace in any one of them corresponds a definite subspace of the same number of dimensions in each of the others. These corresponding subspaces are themselves projectively related.

Now consider a group of corresponding points, one in each of the $m+1$ given n-spaces. These points determine an m-space.

* A *group nucleus* is a neighborhood of the identity; two group nuclei are considered the same if sufficiently small common neighborhoods of the origin are isomorphic.

† Presented to the Society, June 20, 1934.

The given spaces being n-dimensional, we have ∞^n m-spaces so determined, forming an $(m+n)$-dimensional variety $V_{m+n}^{N_{m,n}}$ of order $N_{m,n}$. In this note we are concerned with this variety $V_{m+n}^{N_{m,n}}$. We shall determine, by elementary methods, its order and then write its equations. After stating a few of its properties, we shall, finally, describe the surfaces in which it is met by linear spaces of the proper number of dimensions.

To determine $N_{m,n}$, pass a hyperplane of S_{mn+m+n} through m of the given n-spaces, say $S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m)}$. This hyperplane intersects the remaining n-spaces $S_n^{(m+1)}$ in an $(n-1)$-space $S_{n-1}^{(m+1)}$ to which correspond m $(n-1)$-spaces $S_{n-1}^{(1)}, S_{n-1}^{(2)}, \cdots, S_{n-1}^{(m)}$ in $S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m)}$, respectively, and it intersects the variety $V_{m+n}^{N_{m,n}}$ in a $V_{m+n-1}^{N_{m,n}}$. This $V_{m+n-1}^{N_{m,n}}$ is evidently composed of two varieties, $V_{m+n-1}^{N_{m-1,n}}$ and $V_{m+n-1}^{N_{m,n-1}}$, the former being the locus of the ∞^n $(m-1)$-spaces joining corresponding points of the m projectively related n-spaces $S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m)}$ and the latter the locus of the ∞^{n-1} m-spaces joining corresponding points of the $m+1$ projectively related $(n-1)$-spaces $S_{n-1}^{(1)}, S_{n-1}^{(2)}, \cdots, S_{n-1}^{(m+1)}$. Therefore, the order of $V_{m+n}^{N_{m,n}}$ is

$$N_{m,n} = N_{m-1,n} + N_{m,n-1}.$$

To determine $N_{m-1,n}$ and $N_{m,n-1}$, we proceed in a similar manner and find that

$$N_{m-1,n} = N_{m-2,n} + N_{m-1,n-1},$$
$$N_{m,n-1} = N_{m-1,n-1} + N_{m,n-2}.$$

Continuing this process of derivation until it terminates, we find that we may write, after making the proper substitutions, either

$$N_{m,n} = N_{m-1,n} + N_{m-1,n-1} + \cdots + N_{m-1,0}$$

or

$$N_{m,n} = N_{m,n-1} + N_{m-1,n-1} + \cdots + N_{0,n-1}.$$

Either case yields, since $N_{x,0} = N_{0,x} = 1$, that is, the variety $V_x^{N_{x,0}}$ or $V_x^{N_{0,x}}$ is just a linear x-space, the same result

$$N_{m,n} = \binom{m+n}{m} = \binom{m+n}{n}.$$

This is the order of the variety $V_{m+n}^{N_{m,n}}$.

To derive the equations of the variety, it is convenient to take for the $m+1$ given n-spaces any group of $m+1$ non-intersecting n-spaces of the coordinate simplex of S_{mn+m+n}. We shall take the group of n-spaces whose equations are

$$S_n^{(1)} \quad x_0 : x_1 : \cdots : x_n = u_0 : u_1 : \cdots : u_n,$$
$$x_{n+1} = x_{n+2} = \cdots = x_{mn+m+n} = 0;$$
$$S_n^{(2)} \quad x_0 = x_1 = \cdots = x_n = 0,$$
$$x_{n+1} : x_{n+2} : \cdots : x_{2n+1} = u_0 : u_1 : \cdots : u_n,$$
$$x_{2n+2} : x_{2n+3} : \cdots : x_{mn+m+n} = 0;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n^{(m+1)} \quad x_0 = x_1 = \cdots = x_{mn+m-1} = 0,$$
$$x_{mn+m} : x_{mn+m+1} : \cdots : x_{mn+m+n} = u_0 : u_1 : \cdots : u_n.$$

The coordinates of any point in the m-space joining corresponding points of these $m+1$ n-spaces are given by

$$x_{in+i+j} = t_i u_j, \quad (i = 0, 1, 2, \cdots, m; j = 0, 1, 2, \cdots, n).$$

These are, then, the equations of our $V_{m+n}{}^{N_{m,n}}$ if we regard the t's and u's as variable parameters. Eliminating these parameters we obtain $\binom{m+1}{2}\binom{n+1}{2}$ quadratic equations given by the vanishing of all the two-rowed determinants of the matrix

$$\begin{Vmatrix} x_0 & x_1 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{mn+m} & x_{mn+m+1} & \cdots & x_{mn+m+n} \end{Vmatrix}.$$

We see that the variety $V_{m+n}{}^{N_{m,n}}$ is the common intersection of $\binom{m+1}{2}\binom{n+1}{2}$ quadric hypersurfaces given by the above quadratic equations.

We have already defined our $V_{m+n}{}^{N_{m,n}}$ as the locus of the ∞^n m-spaces determined by corresponding points of $m+1$ projectively related n-spaces. If we interchange m and n, the equations of the variety (and also the result for its order) remain unchanged and therefore we may define $V_{m+n}{}^{N_{m,n}}$ as the locus of

the ∞^m n-spaces joining corresponding points of $n+1$ projectively related m-spaces. Since the projectivity between $m+1$ given n-spaces in an S_{mn+m+n} may be determined by the points of another given n-space, $S_n^{(m+2)}$, that is, from a point of $S_n^{(m+2)}$ one and only one m-space can be constructed incident with the $m+1$ given n-spaces, we see that $V_{m+n}^{N_{m,n}}$ is also the locus of the ∞^n m-spaces incident with $m+2$ given n-spaces. Similarly, it is the locus of the ∞^m n-spaces incident with $n+2$ given m-spaces.

From these definitions, we see that our variety contains a system of ∞^n m-spaces and a system of ∞^m n-spaces. No two m-spaces nor two n-spaces can intersect but each m-space meets each n-space in a point. Through each point of the variety pass one m-space and one n-space.

For $m=n=1$, we have a quadric surface in 3-space. If $m=2$, $n=1$, we have a V_3^3 in 5-space which is the locus of the ∞^1 planes incident with 4 lines and at the same time the locus of the ∞^2 lines incident with 3 planes.

The case $m=1$ and n general is of interest. The variety V_{n+1}^{n+1} in S_{2n+1} is the locus of the ∞^1 n-spaces incident with $n+2$ lines and also the locus of the ∞^n lines incident with 3 n-spaces. Since V_{n+1}^{n+1} is intersected by a general n-space of S_{2n+1} in $n+1$ points, we have the result that there are $n+1$ lines incident with 4 n-spaces given in general positions in S_{2n+1}. For $n=1$, we have the familiar case of the two transversals of 4 general lines in 3-space.

Now the variety $V_{m+n}^{N_{m,n}}$ is intersected by an $S_{m(n+1)}$ of S_{mn+m+n} in a $V_m^{N_{m,n}}$ and by an $S_{n(m+1)}$ in a $V_n^{N_{m,n}}$. Both of these varieties are rational, the one representable upon an S_m and the other representable upon an S_n, respectively. We may assume $m \geqq n$. Then, $V_m^{N_{m,n}}$ is the locus of ∞^n $(m-n)$-spaces and $V_n^{N_{m,n}}$ is the locus of ∞^n points. Let us consider the case $n=2$. The variety $V_{m+2}^{N_{m,2}}$ or $V_{m+2}^{(m+1)(m+2)/2}$ is now the locus of the ∞^2 m-spaces joining corresponding points of $m+1$ given projectively related planes α, β, γ in an S_{3m+2}, and is intersected by a given S_{2m+2} of S_{3m+2} in a rational surface F of order $(m+1)(m+2)/2$. Let the projectivity between these planes be determined by the points of a fourth plane, ϕ. From a point P of ϕ one and only one m-space S_m can be drawn meeting α, β, γ each in a point. We shall take these points for corresponding points in the projectivity. Now S_m meets the given S_{2m+2} in a point P' which is

on F. Thus, we have a one-to-one correspondence between the points of F and those of ϕ.

It can be easily shown that there exist $m(m+1)/2$ points A_i, $(i=1, 2, \cdots, m(m+1)/2)$, in the plane ϕ (or in any plane of S_{2m+2}) from each of which an m-space can be constructed meeting α, β, γ each in a point and S_{2m+2} in a line. Therefore, F has on it $m(m+1)/2$ lines which are the images of A_i in ϕ. If P in ϕ describes a line not passing through any of the points A_i, the m-space S_m describes a V_{m+1}^{m+1} in S_{2m+2} which is met by S_{2m+2} in a rational curve C^{m+1} of order $m+1$ lying on F. The curve C^{m+1} is the locus of the corresponding points P' and is the image of the line described by P. From what has just been said we can see easily that the fundamental curves of representation in ϕ are the ∞^{2m+2} curves of order $m+1$ all passing through the $m(m+1)/2$ points A_i whose images are the $(m+1)(m+2)/2$-ic curves in which the $(2m+1)$-spaces of S_{2m+2} intersect F.

A little calculation shows that the projection of the rational surface F upon a 4-space has $m(m-1)(m^2+7m-6)/8$ improper double points and that its projection upon a 3-space is of class $3m^2$ and rank $2m(m+1)$ and has a double curve whose order is $m(m+1)(m^2+5m-2)/8$, and upon which lie $2m(2m-1)$ pinch points. If we project F from an S_{2m-2} containing $2m-1$ general points of it upon a 3-space, we have for projection a surface F' of order $(m^2-m+4)/2$, class $3m^2$, and rank $2(m^2-m+1)$. Its double curve is of order $m(m^2-1)(m-2)/8$ and on it there are $2(2m-1)(m-2)$ pinch points. The surface F' is representable upon a plane by the ∞^3 $(m+1)$-ic curves all passing through $(m^2+5m-2)/2$ base points. All these results hold for $m \geqq 2$. For $m=2$, F' is a cubic surface whose representation upon a plane by means of the ∞^3 cubic curves through 6 given points is well known. If $m=3$, we have a quintic surface containing a nodal curve of order 3 with 10 pinch points. The fundamental curves of representation in this case are the ∞^3 quartic curves through 11 base points.

THE UNIVERSITY OF CALIFORNIA

A NOTE ON UNITS IN SUPER-CYCLIC FIELDS

BY H. S. VANDIVER

1. *Comparison of Two Known Results Concerning Cyclotomic Units.* Kummer* first showed that if

$$\zeta = e^{2i\pi/l}$$

with l an odd prime, and if η is a unit in $k(\zeta)$ such that

$$\eta \equiv a \pmod{l},$$

where a is a rational integer, then

$$\eta = \rho^l,$$

where ρ is in $k(\zeta)$, provided none of the Bernoulli numbers

$$B_1, B_2, \cdots, B_d, \qquad (d = (l-3)/2), \tag{1}$$

is divisible by l. Kummer's proof of this depended on the fact that under the assumptions mentioned there exists an integer c prime to l such that

$$\eta^c = E_1^{a_1} E_2^{a_2} \cdots E_d^{a_d}. \tag{2}$$

Here

$$E_n = \prod_{i=0}^{d} \epsilon(\zeta^{r^i})^{r^{-2in}},$$

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)\,(1-\zeta^{-1})}\right)^{1/2}.$$

From this we obtain an identity in an indeterminate x by adding a certain multiple of

$$\frac{x^l - 1}{x - 1}.$$

Setting $x = e^v$, taking logarithms and differentiating $2n$ times, $(n = 1, 2, \cdots, d)$, we find, using relations in another paper,†

* Journal für Mathematik, vol. 40 (1850), p. 128.

† Transactions of this Society, vol. 31 (1929), pp. 619–620, relations (4) and (5).

$$a_1 \equiv a_2 \cdots \equiv a_d \equiv 0 \pmod{l},$$

which is the result.

By using a quite different method, Hilbert* gave proof that if

$$\eta \equiv 1 \pmod{\lambda^l}, \qquad \lambda = 1 - \zeta$$

and $k(\zeta)$ is a regular field, then $\eta = \rho^l$.

A field $k(\zeta)$ is said to be regular if and only if l is prime to its class number. It is known that this condition is equivalent to the statement that the set (1) contains no numbers divisible by l.

Comparing the different forms of η in the two statements of Kummer and Hilbert, we note that if $\eta = a + \theta l$, where θ is in $k(\zeta)$, we may write $\theta = b + \lambda\theta_1$, where b is rational, and obtain $\eta = a + lb + \lambda^l\omega$. Now $(a + lb)$ is not necessarily equal to 1, so the two forms are not the same.

Hilbert's proof of the result as stated by him depended on his theory of class-fields. It was reproduced by Landau† who commented on the great length of the proof and the complexity of one of the lemmas involved, that is, the existence of a system of relative fundamental units in a Kummer field.

In the present paper I shall consider further the principles involved in the demonstration of this theorem and give an extension of it involving super-cyclic fields. I shall also consider analogous questions in connection with the cyclotomic field which is not regular. The proofs, in the main, will be merely sketched.

2. *A Theorem Concerning Primary Units in Super-Cyclic Fields.* Furtwängler‡ gave the result that if K contains the field $k(\zeta)$ and if the class number of K be $H = l^h q$, $q \not\equiv 0 \pmod{l}$, and a basis for the Abelian group formed by the qth powers of the ideal classes of K be $C_1, C_2, \cdots, C_e$, then there exists a basis for the singular primary numbers in K, $\omega_1, \omega_2, \cdots, \omega_e$, such that any singular primary number in K may be written in the form $\omega_1^{a_1}\omega_2^{a_2} \cdots \omega_e^{a_e}\alpha^l$. Also, corresponding to any singular primary number belonging to the basis, there is an ideal class C belonging to the basis of the so-called irregular class group.

* Werke, vol. 1, p. 287.

† *Vorlesungen über Zahlentheorie*, vol. 3, p. 240, p. 258.

‡ Mathematische Annalen, vol. 43 (1907), p. 18.

The above shows that if we have the primary units in K, that is, a unit η such that

$$\eta \equiv \gamma^l \pmod{\lambda^l},$$

it follows that if the field K has a class number which is prime to l, then no C exists and therefore no singular primary number. Hence η is an lth power in K.

The above argument can be put in somewhat different form by employing the law of reciprocity

$$\left\{\frac{\alpha}{\beta}\right\} = \left\{\frac{\beta}{\alpha}\right\},$$

where each member denotes an lth power character in K and α is a primary integer in K. As a special case of this we have*

$$\left\{\frac{\omega}{\beta}\right\} = 1,$$

where ω is a singular primary number in K. Let $\beta = \mathfrak{p}^h$, where $\mathfrak{p}$ is a prime ideal in K and h is the class number of K; then the above relation gives

$$\left\{\frac{\omega}{\mathfrak{p}}\right\} = 1$$

for any $\mathfrak{p}$ in K prime to l. From this it follows† that ω is the lth power of the number in K; whence η is also an lth power. We may then state the following theorem.

If an algebraic field K contains a cyclotomic field $k(\zeta)$, $\zeta = e^{2i\pi/l}$, and η is a primary unit in the former field, then η is the lth power of a unit in K provided the class number of K is prime to l.

We now observe that super-cyclic fields exist in which the class number is prime to l. Such a field is a Kummer field defined by ζ and $(\sigma)^{1/l}$, where σ is a unit in $k(\zeta)$ which is not primary. The class number of such a field is prime to l, provided‡ the class number of $k(\zeta)$ is prime to l.

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* Takagi, Journal of the College of Sciences, Tokyo, vol. 44 (1922), p. 26.

† Hilbert, Werke, vol. 1, p. 276.

‡ Pollaczek, Mathematische Zeitschrift, vol. 21 (1924) p. 6.

3. *The Unit E_n not an l^2th Power.* We now consider the units in $k(\zeta)$ when the class number of this field is not prime to l. In this case the integer c in (2) might be divisible by l; in particular one of the E's may be the lth power of the unit in $k(\zeta)$.

We shall now show that if

$$r^{l-1} \not\equiv 1 \pmod{l^2},$$

then

$$E_n \neq \rho^{l^2},$$

where ρ is in $k(\zeta)$. Assuming an equality of this type, and using the same method by which, in a previous paper by the writer, the relations (3) and (3a) were handled,* we obtain the following identity in e^v:

$$E_n^{l-1}(e^v) = (\rho(e^v))^{l^2(l-1)} + X(e^v)(e^{vl} - 1) + lj\frac{e^{vl} - 1}{e^v - 1},$$

where j is a rational integer and $X(e^v)$ is a polynomial in e^v with rational integral coefficients. In this expression, taking logarithms and differentiating $2l$ times, we obtain, using relations (4) and (4a) of the paper last mentioned (p. 620) for $n \neq 1$,

$$\frac{r^{(l-1)(l-n)} - 1}{r^{2l-2n} - 1}\frac{B_l}{2l}(r^{2l} - 1) \equiv 0 \pmod{l^2}.$$

Now, since

$$r^{l-1} \not\equiv 1 \pmod{l^2},$$

then

$$r^{(l-1)(l-n)} - 1$$

is divisible by l but not by l^2, which gives a contradiction since $(r^{2l}-1)$ and B_l/l are prime to l.

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* Loc. cit., p. 617.

A PROBLEM IN ARRANGEMENTS

BY M. H. MARTIN

A problem in dynamics, recently considered by the author, gave rise to the following question.

Let us consider the n^r permutations of n different symbols $e_1, e_2, \cdots, e_n$ taken r at a time with repetitions allowed. *Can a sequence of these symbols be constructed such that each of these n^r permutations is found exactly once as a subsequence of r consecutive symbols in this sequence?*

Thus for $n=10$ and $r=5$ the question is equivalent to asking whether there exists a succession of digits such that every five-place number* is found exactly once among the consecutive digits of the succession. To consider a simpler example, in which the answer is readily given, let us take $n=3$ and $r=2$. Here the $n^r=3^2=9$ permutations are

$$e_1e_1,\ e_1e_2,\ e_1e_3,\ e_2e_1,\ e_2e_2,\ e_2e_3,\ e_3e_1,\ e_3e_2,\ e_3e_3,$$

and a sequence possessing the desired properties is

$$e_1e_3e_3e_2e_3e_1e_2e_2e_2e_1e_1. \tag{1}$$

We now return to the general case and show that the answer to the question in italics above is in the affirmative for any pair of positive integers n and r whatsoever. The proof proceeds by first exhibiting an algorithm, the application of which is then shown to lead to a sequence of the symbols $e_1, e_2, \cdots, e_n$ possessing the desired properties.

From now on a subsequence of r consecutive symbols occurring in a sequence will always be designated as a section of r symbols. Thus $e_3e_2e_3e_1$ is a section of 4 symbols in the sequence (1).

The algorithm in question is built up out of the following three rules.

I. *Each of the first $r-1$ symbols is chosen equal to e_1.*

II. *The symbol a_m to be added to the sequence*

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* By five-place numbers we mean, of course, those like 31342, 41231 etc., including those, however, such as 00123, 00005, etc.

$$(2)\qquad a_1a_2\cdots a_{r-1}a_r\cdots a_{m-r+1}\cdots a_{m-1},\qquad a_1=a_2=\cdots=a_{r-1}=e_1,\quad m\geqq r,$$

where the a's stand for the e's in a certain order, is the e_i with the greatest subscript consistent with the requirement that the section $a_{m-r+1}\cdots a_{m-1}a_m$ duplicate no previously occurring section of r symbols in (2).

III. *Rule* II *is first applied for $m=r$ (in which case $a_m=a_r=e_m$) and is then applied repeatedly until a further application is impossible.*

At this point the reader is urged to verify that the sequence (1) has been constructed by use of this algorithm.

Suppose that the application of Rule I followed by repeated applications of Rule II yields a certain sequence

$$(3)\qquad a_1a_2\cdots a_{r-1}a_r\cdots a_{p-r}a_{p-r+1}\cdots a_{p-1},\qquad a_1=a_2=\cdots=a_{r-1}=e_1,\quad p>r,$$

of the e's. When may Rule II be applied once more to add another symbol, say a_p, to (3)? Before answering this question in the lemmas below we must point out that no two sections of r symbols each in (3) are identical. This fact will be used in the proof of the following lemma.

LEMMA 1. *If at least one of the symbols in the section $a_{p-r+1}\cdots a_{p-1}$ of* (3) *differs from e_1, Rule* II *may be applied once more to add another symbol, say a_p, to* (3).

In order to prove this lemma, let us count the number of times the section $a_{p-r+1}\cdots a_{p-1}$ of $r-1$ symbols in (3) has occurred prior to its occurrence in the right-hand end position. If it has never occurred prior to its occurrence in the right-hand end position, we are able to apply Rule II once more and add another symbol $a_p=e_n$ to (3). Clearly, if it has occurred j, $(j<n)$, times prior to its occurrence in the right-hand end position, Rule II may be applied once more to add $a_p=e_{n-j}$. In order to complete the proof of the lemma we show that under the hypothesis of the lemma (namely, that at least one of the symbols in the section $a_{p-r+1}\cdots a_{p-1}$ differs from e_1) the number of times the section $a_{p-r+1}\cdots a_{p-1}$ occurs prior to its occurrence in the right-hand end position is always less than n. For, suppose the contrary were true. The section $a_{p-r+1}\cdots a_{p-1}$ would

then occur at least $n+1$ times in (3). Since at least one of the symbols contained in it differs from e_1, it cannot occupy the left-hand end position in (3) and hence is preceded by at least one symbol at each of its occurrences. On adjoining this preceding symbol to $a_{p-r+1} \cdots a_{p-1}$ at each of its occurrences we conclude that the sequence (3) contains the following $n+1$ sections of r symbols each:

$$b_i a_{p-r+1} \cdots a_{p-1}, \qquad (i = 1, 2, \cdots, n+1), \tag{4}$$

where b_i stands for one of the e's. Since there are only n of the e's, at least two of the sections (4) must be identical and this is a contradiction to the property of the sequence (3) pointed out above when we introduced this lemma.

Obviously Rule II can only be applied a finite number of times. To answer the question as to when it can no longer be applied, we shall prove the following lemma.

LEMMA 2. *In order that Rule* II *can no longer be applied to add another symbol to* (3) *it is necessary and sufficient that each of the symbols in the section* $a_{p-r}a_{p-r+1} \cdots a_{p-1}$ *of* (3) *is equal to* e_1.

To show the necessity of the condition suppose Rule II can not be applied to (3). Then, first of all, it follows from Lemma 1 that each of the symbols $a_{p-r+1}, \cdots, a_{p-1}$ is equal to e_1. Furthermore, the sequence (3) must contain the following n sections of r symbols each:

$$e_1 e_1 \cdots e_1 e_{n-j}, \qquad (j = 0, 1, \cdots, n-1); \tag{5}$$

otherwise Rule II could be applied to (3). The section $j=0$ in (5) occupies the left-hand end position in (3), the remaining sections $j=1, 2, \cdots, n-1$ being distributed from left to right in (3) with increasing j. The section $j=n-1$ in (5) is composed exclusively of e_1's and occupies the right-hand end position in (3). For, suppose it did not. Suppress those symbols which would then lie to the right of it in (3). The sequence so obtained contains all of the sections in (5) and each of the last $r-1$ symbols in it is equal to e_1. Hence Rule II cannot be applied to add another symbol and this is a contradiction to the assumption that certain symbols lie to the right of the section $j=n-1$ of (5) in the sequence (3). Therefore our assumption that the sec-

tion $j=n-1$ in (5) does not occupy the right-hand end position is false.

To show the sufficiency of the condition suppose that each of the symbols in the section $a_{p-r}a_{p-r+1}\cdots a_{p-1}$ of (3) is equal to e_1. From the nature of Rule II it follows that each of the sections in (5) occurs in (3). Since each of the last $r-1$ symbols in (3) is equal to e_1, Rule II cannot be applied to add another symbol.

Let us suppose that we have constructed a sequence

$$a_1a_2\cdots a_{r-1}a_r\cdots a_{N-r+1}\cdots a_N, \quad a_j = a_{N-j} = a_N = e_1, \quad (j = 1, 2, \cdots, r-1), \tag{6}$$

according to our algorithm. With respect to this sequence we shall prove the following theorem.

THEOREM. *An arbitrarily given permutation of the symbols $e_1, e_2, \cdots, e_n$ taken r at a time with repetitions allowed occurs exactly once as a section of r symbols in the sequence* (6).

That a given permutation can occur only once as a section of r symbols in (6) is an immediate consequence of Rule II in the algorithm. That a given permutation actually occurs as a section of r symbols in (6) is taken up next. First of all, since the last r symbols in (6) are each equal to e_1, the sequence (6) contains each of the sections (5). (See the sufficiency proof of Lemma 2.) The section $j=0$ in (5) is the only one that occupies the left-hand position in (6). Since the section $j=n-1$ in (5) contains two sections of $r-1$ consecutive e_1's, a section of $r-1$ consecutive e_1's occurs n times in (6) in addition to its occurrence there in the left-hand end position. Now no two sections of r symbols can be identical in (6). Hence each of the sections

$$e_je_1e_1\cdots e_1, \qquad (j = 1, 2, \cdots, n), \tag{7}$$

of r symbols each, occurs in (6). From the nature of Rule II in the algorithm and the presence of the sections (7) in (6) it follows that the sections

$$e_je_1e_1\cdots e_1e_k, \qquad (j, k = 1, 2, \cdots, n), \tag{8}$$

of r symbols each, also occur in (6).

Let $b_1b_2b_3\cdots b_{r-1}b_r$ denote an arbitrary permutation of

$e_1, e_2, \cdots, e_n$ with repetitions of a symbol allowed. We show that this permutation can be found as a section of r symbols in (6). The proof of this fact has been given in the preceding paragraph if $b_2 = b_3 = \cdots = b_{r-1} = e_1$ and we rule this case out in the following. In order to show that $b_1 b_2 b_3 \cdots b_{r-1} b_r$ occurs as a section of r symbols in (6) it is clearly sufficient to show that $b_1 b_2 b_3 \cdots b_{r-1} e_1$ occurs there as a section of r symbols. Now suppose that $b_1 b_2 b_3 \cdots b_{r-1} e_1$ does not occur as a section of r symbols in (6). Since the section $b_2 b_3 \cdots b_{r-1} e_1$ contains, by hypothesis, a symbol different from e_1, it cannot occupy the left-hand end position in (6). Hence it can occur at most $n-1$ times in (6); otherwise the section $b_1 b_2 b_3 \cdots b_{r-1} e_1$ would have to occur in (6) contrary to our hypothesis. Since the section $b_2 b_3 \cdots b_{r-1} e_1$ occurs at most $n-1$ times in (6), the section $b_2 b_3 \cdots b_{r-1} e_1 e_1$ cannot occur in (6). Two possibilities are now open, namely:

(a) The section $b_3 \cdots b_{r-1} e_1 e_1$ contains no symbol different from e_1,

(b) The section $b_3 \cdots b_{r-1} e_1 e_1$ contains a symbol different from e_1.

If (a) presents itself, the conclusion that the section $b_2 b_3 \cdots b_{r-1} e_1 e_1$ cannot occur in (6) is a contradiction to the existence of the sections (7) in (6). Hence, in this case, our assumption that the section $b_1 b_2 b_3 \cdots b_{r-1} e_1$ does not occur in (6) was false.

If (b) presents itself, a repetition of our earlier reasoning is used to show that the section $b_3 b_4 \cdots b_{r-1} e_1 e_1 e_1$ cannot occur in (6) and the discussion is repeated on the basis that this section cannot occur in (6), etc. Barring any earlier arrival to a contradiction of our results in (7) we must eventually reach the contradiction that the section $b_{r-1} e_1 e_1 \cdots e_1$ of r symbols cannot occur in (6).

The question raised at the beginning of this paper is now disposed of and we close by pointing out an obvious extension to our results. Consider, as before, the n^r permutations of n different symbols $e_1, e_2, \cdots, e_n$ taken r at a time with repetitions allowed. It is possible to construct a sequence of these symbols such that each of the n permutations is found exactly k times as a section of r symbols in the sequence

To show this let us first take $k=2$. Consider two sequences of the e's each identical with the sequence (6). From one of these

sequences remove the first $r-1$ of the e_1's occurring in the left-hand end position and place this truncated sequence on the same horizontal line with the other and to its right. The resulting sequence of the e's obviously contains each of the n^r permutations exactly twice.

The case of an arbitrary k is then readily disposed of by successive applications of this process.

TRINITY COLLEGE

CRITERIA FOR THE IRREDUCIBILITY OF POLYNOMIALS*

BY LOUIS WEISNER

1. *Introduction.* If a polynomial with integral coefficients is reducible in the field of rational numbers, the task of decomposing it into the product of irreducible polynomials may be expected to involve a great deal of numerical work, commensurate with the degree and coefficients of the polynomial, such as is required by Kronecker's method. But when it is merely required to know whether or not the polynomial is reducible, the amount of labor required by Kronecker's method is altogether too great. As a polynomial is completely determined by a sufficiently extended table of values, these values should suffice to determine the reducibility or irreducibility of the polynomial. We can hardly expect to establish the *reducibility* of a polynomial of degree n, with fewer than $n+1$ entries in its table of values. For this reason criteria establishing the reducibility of a polynomial are unknown. No such criteria are established in the present paper. On the other hand, *one* entry in the table of values of a polynomial may be sufficient to establish its *irreducibility*. The present paper is concerned with criteria of this sort.

One such criterion is available:† if for a sufficiently large integer h, $f(h)$ is a prime, where $f(x)$ is a polynomial with inte-

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† See P. Stäckel, Journal für Mathematik, vol. 148 (1918), p. 109; Pólya and Szegö, *Aufgaben und Lehrsätze*, vol. 2, p. 137, Ex. 127.

gral coefficients, then $f(x)$ is irreducible.* The applicability of this theorem is clearly limited, for it is well known that an irreducible polynomial may represent no prime. Moreover, the first prime represented by a polynomial may be large, and considerable fruitless calculations may be expended in discovering it.

The criteria described in §3 have a wider range of applicability. The sense of these criteria is that, subject to certain conditions, a polynomial is irreducible if it represents the integer $\pm kp^m$, (p prime), where k is relatively small. These criteria are well adapted to establishing the irreducibility of numerical polynomials and lead to the construction of large classes of irreducible polynomials.

2. *Irreducibility Determined by Leading and Final Coefficients.*

THEOREM 1. *Let L and M be lower and upper bounds, respectively, of the absolute values of the roots of a reducible polynomial*

$$A(x) = \sum_{v=0}^{n} a_v x^{n-v}, \qquad (n \geqq 2,\ a_0 \neq 0),$$

with integral coefficients, and suppose that

$$|a_n| = kp^m, \qquad (k \geqq 1,\ m \geqq 1),$$

where p is a prime which does not divide a_{n-1} if $m > 1$.

A. *If $L \geqq 1$, then $k \geqq L$.* B. *If $M \geqq 1$, then $p^m \leqq |a_0| M^{n-1}$.*

As $A(x)$ is reducible,

$$A(x) = B(x)C(x) = \sum_{v=0}^{r} b_v x^{r-v} \sum_{v=0}^{s} c_v x^{s-v},$$

$$(1 \leqq r \leqq n-1;\ 1 \leqq s \leqq n-1),$$

the b's and c's being integers. We have

$$a_{n-1} = b_{r-1}c_s + b_r c_{s-1},$$
$$kp^m = |a_n| = |b_r||c_s|.$$

* Except where the contrary is explicitly stated, the terms *reducible* and *irreducible* will be understood to apply to the field of rational numbers throughout this paper.

As a_{n-1} is not divisible by p if $m>1$, one of the integers b_r, c_s is not divisible by p. If b_r is not divisible by p, c_s is divisible by p^m. If $m=1$, at least one of the integers b_r, c_s is divisible by p, and we shall suppose that c_s has this property. In both cases c_s is divisible by p^m. Hence

$$|c_s| \geqq p^m, \qquad |b_r| \leqq k.$$

Denoting the roots of $B(x)$ by $\beta_1, \cdots, \beta_r$, we have

$$|\beta_1| \cdots |\beta_r| = \left|\frac{b_r}{b_0}\right| \leqq |b_r| \leqq k.$$

Hence, as the roots of $B(x)$ are roots of $A(x)$,

$$k \geqq L^r,$$

from which the first part of the theorem follows.

Denoting the roots of $C(x)$ by $\gamma_1, \cdots, \gamma_s$, we have

$$|\gamma_1| \cdots |\gamma_s| = \left|\frac{c_s}{c_0}\right| \geqq \frac{p^m}{|c_0|} \geqq \frac{p^m}{|a_0|}$$

as c_0 is a divisor of a_0. Hence

$$p^m \leqq |a_0| M^s,$$

from which the second part of the theorem follows.

The rational roots, and hence the linear factors, of a polynomial with integral coefficients, are readily determined by elementary methods. If $A(x)$ is reducible, but has no linear factors, $2 \leqq r \leqq n-1$, $2 \leqq s \leqq n-2$. Hence we have the following theorem.

THEOREM 2. *With the notation and hypothesis of Theorem* 1, *if $A(x)$ is reducible but has no linear factor, then*

$$k \geqq L^2, \qquad p^m \leqq |a_0| M^{n-2}.$$

As an illustration, consider the polynomial

$$x^n + x \pm p^m, \qquad (p^m > 2).$$

Its roots are easily shown to be outside the unit circle. It follows from Theorem 1A, with $k=1$, that the polynomial is irreducible if p^m is any prime-power integer >2.

3. Irreducibility Determined by Integers Represented by a Polynomial.

THEOREM 3. *Let M be an upper bound of the absolute values of the roots of a polynomial $A(x)$ of degree n with integral coefficients and leading coefficient $a_0 \neq 0$; and let h be an integer such that*

$$|h| \geqq M + 1, \qquad |A(h)| = kp^m,$$

where p is a prime which does not divide $A'(h)$ if $m > 1$. If $A(x)$ is reducible,

$$k \geqq |h| - M, \qquad p^m \leqq |a_0| (|h| + M)^{n-1}.$$

If $A(x)$ is reducible but has no linear factor,

$$k \geqq (|h| - M)^2, \qquad p^m \leqq |a_0| (|h| + M)^{n-2}.$$

Evidently, $A(x)$ and $A(x+h) = a_0x^n + \cdots + A'(h)x + A(h)$ are simultaneously reducible or irreducible. If ρ is a root of $A(x+h)$, $\rho+h$ is a root of $A(x)$, so that $|\rho+h| \leqq M$. Hence

$$|\rho| = |h - (\rho + h)| \geqq |h| - |\rho + h| \geqq |h| - M,$$

$$|\rho| \leqq |h| + M.$$

As $|h| - M$ and $|h| + M$ are lower and upper bounds, respectively, of the absolute values of the roots of $A(x+h)$, the theorem follows from those of §2.

As an illustration, consider the polynomial

$$A(x) = x^6 - 2x^4 + x^3 + x^2 - x - 3,$$

which represents only multiples of 3, so that Stäckel's criterion (§1) is inapplicable. Here we may take $M = 2$. We find that

$$A(-5) = 3 \cdot 4759,$$

and 4759 is a prime. As the polynomial has no linear factor, and the inequality

$$k \geqq (|h| - M)^2$$

is violated, the polynomial is irreducible.

If M is large, the application of Theorem 2 involves computations with large figures. The next theorem may then be found more practicable. Let

$$A(x, y) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu} y^\nu$$

be the homogeneous polynomial corresponding to

$$A(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu}.$$

The homogeneous polynomial corresponding to $A'(x)$ is denoted by $A'(x, y)$.

THEOREM 4. *Let L be a lower bound of the absolute values of a polynomial $A(x)$ with integral coefficients; and let t and u be integers such that*

$$|u|L - |t| \geqq 1, \qquad |A(t, u)| = kp^m,$$

where p is a prime which does not divide $A'(t, u)$ if $m > 1$. If $A(x)$ is reducible,

$$k \geqq |u|L - |t|.$$

If $A(x)$ is reducible but has no linear factor,

$$k \geqq (|u|L - |t|)^2.$$

It follows, as in the proof of Theorem 3, that $|u|L - |t|$ is a lower bound of the absolute values of the roots of the polynomial

$$A(x + t, u) = a_0 x^n + \cdots + A'(t, u)x + A(t, u),$$

which is reducible if $A(x)$ is. The theorem follows by those of §2.

4. *Certain Classes of Irreducible Polynomials.*

THEOREM 5. *If $f(x)$ is a polynomial with integral coefficients, which has a rational root h, and k is a fixed positive integer, the polynomial $f(x) \pm kp$ is irreducible for all sufficiently large primes p. If, in addition, $f'(h) \neq 0$, the polynomial $f(x) \pm kp^m$ is irreducible for all sufficiently large prime-power integers p^m.*

Let $h = t/u$, where t and u are integers and $u \geqq 1$, and let n be the degree of $f(x)$. If

$$g(x) = u^n f\left(x + \frac{t}{u}\right),$$

then

$$u^n f\left(x+\frac{t}{u}\right) \pm ku^n p^m = g(x) \pm ku^n p^m,$$

and $g(0)=0$. It is therefore sufficient to prove the theorem for $h=0$.

If the polynomial

$$A(x) = f(x) \pm kp^m = a_0 x^n + \cdots + a_{n-1}x \pm kp^m, \quad (a_{n-1} \neq 0),$$

has a root whose absolute value is $\leqq k$, then

$$kp^m \leqq |a_0| k^n + \cdots + |a_{n-1}| k.$$

Hence, if p^m is a prime-power integer greater than

$$|a_0| k^{n-1} + \cdots + |a_{n-1}|,$$

all the roots of $A(x)$ are $>k$ in absolute value. It follows from Theorem 1A that $A(x)$ is irreducible unless $m>1$ and p is a divisor of a_{n-1}.

To treat this case, let p be a prime factor of a_{n-1}, p^μ the highest power of p that divides a_{n-1}, and suppose that $A(x)$ is reducible. With the notation of the proof of Theorem 1, we have

$$b_{r-1}c_s + b_r c_{s-1} = a_{n-1},$$

$$b_r c_s = \pm kp^m.$$

At least one of the integers b_r, c_s is not divisible by $p^{\mu+1}$. If b_r has this property,

$$|b_r| = k_1 p^\nu, \qquad (0 \leqq \nu \leqq \mu),$$

where k_1 is a divisor of k. Hence

$$|b_r| \leqq k |a_{n-1}|.$$

At least one root β of $B(x)$ therefore satisfies the inequality

$$|\beta| \leqq \left(\left|\frac{b_r}{b_0}\right|\right)^{1/r} \leqq (|ka_{n-1}|)^{1/r} \leqq |ka_{n-1}|.$$

On the other hand, by choosing m sufficiently large, the absolute value of every root of $A(x)$ may be made arbitrarily large. Hence $A(x)$ is irreducible for sufficiently large values of m.

That the condition $f'(h) \neq 0$ if $m > 1$ is necessary is shown by the example $x^n - p^m$, which is reducible if p is any prime and m any multiple of n, $(n > 1)$.

THEOREM 6. *Let*

$$A(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu}$$

be a polynomial with integral coefficients. If a_n is a power of a prime which does not divide a_{n-1}, and if

$$0 < a_0 < a_1 \cdots < a_{n-1} < a_n,$$

then $A(x)$ is irreducible.

The stated inequalities imply that the absolute value of every root of $A(x)$ is >1.* The theorem follows from Theorem 1, with $k = 1$.

5. *Validity of Preceding Theorems in Other Fields.* There are fields besides the field of rational numbers for which the preceding theorems are valid. Let R be an algebraic field of class number 1. The first five theorems are valid if we understand reducibility to pertain to R and *integer* to mean integer of R. Theorem 6 is valid, with a similar understanding, if the coefficients of $A(x)$ are real numbers.

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* See G. Eneström, *Härledning af en allmän formel . . .*, Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar, vol. 50 (1893), p. 405–415. French translation in Tôhoku Mathematical Journal, vol. 18 (1920), p. 34. An elegant proof of Eneström's theorem is given by A. Hurwitz, Tôhoku Mathematical Journal, vol. 4 (1914), p. 89.

ON AN EXPANSION OF THE REMAINDER IN THE GAUSSIAN QUADRATURE FORMULA*

BY J. V. USPENSKY

1. *Introduction.* The Gaussian quadrature formula

$$(1) \qquad \int_0^1 f(x)dx = A_1 f(x_1) + A_2 f(x_2) + \cdots + A_n f(x_n),$$

in which $x_1, x_2, \cdots, x_n$ are roots of Legendre's polynomial

$$P_n(x) = \frac{d^n x^n (x-1)^n}{dx^n},$$

and

$$A_i = \int_0^1 \frac{P_n(x)}{(x - x_i)P_n'(x_i)}\,dx, \quad (i = 1, 2, \cdots, n),$$

is exact in case $f(x)$ is an arbitrary polynomial of degree not exceeding $2n-1$. Otherwise the formula (1) is only approximate, and the difference between its left and right hand sides represents the error or remainder term which will be denoted in what follows by R_n. The expression of this remainder, obtained, if I am not mistaken, for the first time by A. A. Markoff in 1884, is well known. In this article I shall prove that the remainder in the Gaussian formula can be expanded into a series possessing all the properties of the classical Euler-Maclaurin expansion. This is a noteworthy fact, equally important from the theoretical and from the practical point of view.

2. *Expression of R_n.* In what follows we shall adopt E. Nörlund's definition of the Bernoullian polynomial $B_n(x)$ of order n; and we shall define the periodic function $\overline{B}_n(x)$ by the equations

$$\overline{B}_n(x) = B_n(x), \text{ for } 0 \leqq x < 1;$$

$$\overline{B}_n(x+1) = \overline{B}_n(x), \text{ for all } x.$$

With these notations, we have, for $0 \leqq \theta \leqq 1$,

* Presented to the Society, June 20, 1934.

$$f(\theta) = \int_0^1 f(x)dx + \sum_{s=1}^{2n} \frac{B_s(\theta)}{s!}\{f^{(s-1)}(1) - f^{(s-1)}(0)\} - \int_0^1 \frac{\overline{B}_{2n}(\theta - t)}{(2n)!} f^{(2n)}(t)dt.$$

Taking here $\theta = x_1, x_2, \cdots, x_n$, multiplying the resulting equations by $A_1, A_2, \cdots, A_n$, and adding them, we get

$$\sum_{i=1}^{n} A_i f(x_i) = \int_0^1 f(x)dx + \frac{1}{(2n)!}\sum_{i=1}^{n} A_i B_{2n}(x_i)\{f^{(2n-1)}(1) - f^{(2n-1)}(0)\} - \int_0^1 \frac{f^{(2n)}(t)}{(2n)!}\sum_{i=1}^{n} A_i \overline{B}_{2n}(x_i - t)dt, \tag{2}$$

since for $s = 1, 2, \cdots, 2n-1$,

$$\sum_{i=1}^{n} A_i B_s(x_i) = \int_0^1 B_s(x)dx = 0.$$

For brevity, we shall use the notations

$$\overline{B}_p(\theta - t) - B_p(\theta) = F_p(\theta, t), \qquad \sum_{i=1}^{n} A_i F_p(x_i, t) = G_p(t).$$

Then equation (2) yields

$$R_n = \frac{1}{(2n)!}\int_0^1 G_{2n}(t) f^{(2n)}(t)dt. \tag{3}$$

3. *The Function $G_p(t)$.* It follows immediately from the definition of the function $G_p(t)$ that $G_p(0) = G_p(1) = 0$. Moreover

$$G_{2s}(1 - t) = G_{2s}(t), \qquad G_{2s-1}(1 - t) = -G_{2s-1}(t). \tag{4}$$

The proof of these relations essentially depends upon the fact that the numbers $x_1, x_2, \cdots, x_n$ are symmetrically located with respect to $\frac{1}{2}$, so that, if these numbers are arranged in increasing order, $x_{n-i+1} = 1 - x_i$ and at the same time $A_{n-i+1} = A_i$. We have

$$G_p(1-t) = \sum_{i=1}^{n} A_i\{\overline{B}_p(x_i+t) - B_p(x_i)\}$$

$$= \sum_{i=1}^{n} A_{n-i+1}\{\overline{B}_p(x_{n-i+1}+t) - B_p(x_{n-i+1})\}$$

$$= \sum_{i=1}^{n} A_i\{\overline{B}_p(1+t-x_i) - B_p(1-x_i)\} = (-1)^p G_p(t),$$

which amounts to the two relations written above.

Similarly,

$$\sum_{i=1}^{n} A_i B_{2s-1}(x_i) = \sum_{i=1}^{n} A_{n-i+1}B_{2s-1}(x_{n-i+1}) = \sum_{i=1}^{n} A_i B_{2s-1}(1-x_i)$$

$$= -\sum_{i=1}^{n} A_i B_{2s-1}(x_i),$$

whence

$$\sum_{i=1}^{n} A_i B_{2s-1}(x_i) = 0,$$

so that $G_{2s-1}(t)$ can be written in the simple form

$$G_{2s-1}(t) = \sum_{i=1}^{n} A_i \overline{B}_{2s-1}(x_i - t).$$

Since

$$\overline{B}_n'(x) = n\overline{B}_{n-1}(x),$$

it follows from the last expression for $G_{2s-1}(t)$ that

$$G_{2s}'(t) = -2sG_{2s-1}(t),$$

$$G_{2s}''(t) = 2s(2s-1)\left[G_{2s-2}(t) + \sum_{i=1}^{n} A_i B_{2s-2}(x_i)\right]. \tag{5}$$

Furthermore

$$G_{2s+1}''(t) = 2s(2s+1)G_{2s-1}(t). \tag{6}$$

4. *Sign of* $G_{2s}(t)$. Our main purpose is to show that, for $s \geqq n$, functions $G_{2s}(t)$ do not change sign in the interval $0<t<1$. To this end let β_s and α_s represent the number of distinct roots of the equations $G_{2s}(t)=0$ and $G_{2s-1}(t)=0$ in the interval $0<t<1$, respectively. The second of the relations (4) shows that

$G_{2s-1}(1/2)=0$; hence $\alpha_s \geqq 1$. It follows from the first of the relations (5) and Rolle's theorem that $\beta_s+1 \leqq \alpha_s$, because $G_{2s}(0)=G_{2s}(1)=0$. Again, using (6) and applying Rolle's theorem twice, we get $\alpha_s \leqq \alpha_{s-1}$, so that, for $s \geqq n$, $\beta_s+1 \leqq \alpha_n$. But if $0 \leqq t \leqq 1$, we have

$$G_{2n-1}(t) = \sum_{i=1}^{n} A_i \overline{B}_{2n-1}(x_i - t)$$

$$= \sum_{i=1}^{n} A_i B_{2n-1}(x_i - t) + (2n-1) \sum_{x_i \leq t} A_i (x_i - t)^{2n-2},$$

where the second sum in the right member contains only terms in which $x_i \leqq t$. On the other hand, we have

$$\sum_{i=1}^{n} A_i B_{2n-1}(x_i - t) = \int_0^1 B_{2n-1}(x-t)dx$$

$$= \frac{1}{2n}\{B_{2n}(1-t) - B_{2n}(-t)\} = -t^{2n-1},$$

because $B_{2n-1}(x-t)$ is a polynomial in x of degree $2n-1$. It follows that $G_{2n-1}(t)$ differs only by a constant factor from the function

$$R_0(t) = \frac{t^{2n-1}}{2n-1} - \sum_{x_i \leq t} A_i (x_i - t)^{2n-2},$$

which represents the remainder in the Gaussian formula applied to the function defined by the equations

$$f(x) = (x-t)^{2n-2}, \qquad \text{if } x \leqq t,$$

$$f(x) = 0, \qquad \text{if } x > t.$$

5. *Fundamental Lemma. The equation $R_0(t)=0$ has one and only one root in the interval $0<t<1$.*

PROOF. Let

$$R_k(t) = \frac{(-1)^k t^{2n-k-1}}{2n-k-1} - \sum_{x_i \leq t} A_i (x_i - t)^{2n-k-2}$$

for $k=0, 1, 2, \cdots, 2n-2$. The functions $R_0(t), R_1(t), \cdots, R_{2n-3}(t)$ are evidently continuous, but

$$R_{2n-2}(t) = t - \sum_{x_i \leq t} A_i$$

is discontinuous at $x_1, x_2, \cdots, x_n$. By the fundamental property of the Gaussian formula, $R_k(1) = R_k(0) = 0$. On the other hand, $R_k'(t) = -(2n-k-2)R_{k+1}(t)$, for $k = 0, 1, 2, \cdots, 2n-3$. Hence, if N_k is the number of distinct roots of the equation $R_k(t) = 0$, we shall have, by Rolle's theorem, $N_k + 1 \leqq N_{k+1}$ for $k = 0, 1, 2, \cdots, 2n-4$. Hence $N_{2n-3} \geqq N_0 + 2n - 3$. But $N_{2n-3} + 1$ cannot exceed the number of variations of sign of $R_{2n-2}(t)$ when t increases from 0 to 1. Let this number be denoted by N_{2n-2}. Then first, $N_0 + 2n - 2 \leqq N_{2n-2}$; and, second, $N_{2n-2} \leqq 2n - 1$. For, $R_{2n-2}(t)$ can change sign not more than once in each of the $n-1$ intervals (x_i, x_{i+1}), $(i = 1, 2, \cdots, n-1)$, and also possibly at $t = x_1, x_2, \cdots, x_n$. Since $N_0 = \alpha_k \geqq 1$, the inequality $N_0 + 2n - 2 \leqq 2n - 1$ shows that $N_0 = \alpha_n = 1$.

6. *Expansion of* R_n. Since $\alpha_n = 1$, the inequality $\beta_s + 1 \leqq \alpha_n = 1$ established for $s \geqq n$ shows that $\beta_s = 0$, that is, $G_{2s}(t)$ *does not change its sign in the interval* $0 < t < 1$, if $s \geqq n$. After this fundamental point has been established, it suffices to use the formula

$$G_{2s}''(t) = 2s(2s-1)\left[G_{2s-2}(t) + \sum_{i=1}^{n} A_i B_{2s-2}(x_i)\right],$$

and to apply repeatedly integration by parts to the integral in (3) in order to arrive at the following expansion of R_n:

$$(7) \quad R_n = \sum_{s=0}^{k-1} c_s \left\{ f^{(2n+2s-1)}(1) - f^{(2n+2s-1)}(0) \right\} + c_k f^{(2n+2k)}(\xi),$$

where

$$c_s = -\frac{1}{(2n+2s)!} \sum_{i=1}^{n} A_i B_{2n+2s}(x_i) = -\frac{\gamma_{2n+2s}}{(2n+2s)!},$$

and where ξ is an unknown number between 0 and 1. To show that the expansion (7) possesses all the properties of the Euler-Maclaurin expansion, it suffices to prove that numbers γ_{2n}, γ_{2n+2}, $\gamma_{2n+4}, \cdots$ alternate in sign. To this end, we remark first that

$$\int_0^1 G_{2n+2s}(t)dt = -\gamma_{2n+2s},$$

and, second, that the sign of $G_{2n+2s}(t)$ in the interval (0, 1) is the same as the sign of

$$G''_{2n+2s}(0) = (2n + 2s)(2n + 2s - 1)\gamma_{2n+2s-2}.$$

Hence γ_{2n+2s} and $\gamma_{2n+2s-2}$ have opposite signs, which was to be proved.

The coefficient c_0 is positive. For, since for small values of t the derivative $G'_{2n}(t)$ is greater than 0, $G_{2n}(t)$ will be positive for $0<t<1$; hence $\gamma_{2n}<0$ and $c_0>0$. Thus in general $(-1)^s c_s>0$. The expansion (7) is especially useful in numerical applications when all derivatives of an even order $\geqq 2n$ have the same sign in (0, 1). For then, if we retain a certain number of terms in (7), the error in R_n will in absolute value be less than the first neglected term and will have the same sign as this term.

7. *Values of* c_n. The simplest way of finding the general expressions of $c_0, c_1, c_2, \cdots$ consists in taking in the Gaussian formula successively $f(x)=P_nP_n$, P_nP_{n+2}, $P_nP_{n+4}, \cdots$. Then $c_0, c_1, c_2, \cdots$ are one by one determined by a set of linear equations. While this method is theoretically simple, nevertheless the actual calculation is very laborious. Here are the expressions of c_0, c_1, c_2:

$$c_0=\left\{\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n}\right\}^2 \frac{1}{(2n+1)!},$$

$$c_1=-\frac{n(4n^2+5n-2)}{24(n+1)(2n-1)(2n+3)}c_0,$$

$$c_2=\frac{n(112n^6+384n^5-151n^4-1184n^3-105n^2+635n-156)}{2880(n+1)(n+2)(2n-3)(2n-1)^2(2n+3)(2n+5)}c_0.$$

For particular values of n approximate values of the following coefficients $c_3, c_4, \cdots$ can be found without excessive labor by another method.

STANFORD UNIVERSITY

ON THE THEORY OF RESIDUES OF POLYGENIC FUNCTIONS*

BY V. C. POOR

1. *Introduction.* In polygenic function theory we are interested in a sub-class of the class of all functions, such that

$$f(z) = f(\xi, \zeta),$$

where ξ and ζ are complex variables; that is, such that when ξ and ζ are assigned $f(z)$ is known. The particular sub-class to which we restrict ourselves is the class such that ζ is always the conjugate of ξ, or

$$f(z) = f(z, \bar{z}).$$

For a brief outline of this subject and a quite complete bibliography one should consult the paper by Hedrick† in this Bulletin.

It is the purpose of this paper to generalize the definitions for residues of polygenic functions previously given‡ and to extend the theory. Incidentally in the process, the circulation theorems§ are generalized; a theorem on residues of regular functions is obtained, while the theory is applied to the large class of functions defined by a Laurent series.

2. *The Definitions for Residues.* If we surround the point $z=a$ by a circle O, center at a and radius r, then the residue R_z of $f(z)$ is defined by the equation,‖

$$R_z = \lim_{r \to 0} \frac{1}{2\pi i} \int_O f(z)dz; \tag{1}$$

while the residue $R_{\bar{z}}$, which is of equal importance, is

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* Presented to the Society, December 27, 1933.

† E. R. Hedrick, *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933), pp. 75–96.

‡ V. C. Poor, *Residues of polygenic functions*, Transactions of this Society, vol. 32 (1930), pp. 216–222.

§ Poor, loc. cit. Calugaréano, (Thesis), *Sur les fonctions polygènes d'une variable complexe*, 1928, p. 11.

‖ Poor, loc. cit., §1.

$$R_2 = \lim_{r \to 0} \frac{-1}{2\pi i} \int_O f(z)d\bar{z}; \tag{2}$$

the total residue is then, by definition,

$$R = R_1 + R_2. \tag{3}$$

THEOREM 1. *In the definitions for the residues of a polygenic function, the contour of integration may be replaced by any arbitrary contour C, providing in the limit the area bounded by C contracts to the point a.*

In this theorem we are interested in a point a of a domain D at which R_1 exists. We wish to prove then, when the circle of radius r is replaced by an arbitrary contour C, that

$$\lim_{C \to 0} \frac{1}{2\pi i} \int_C f(z)dz$$

exists and is equal to R_1 if C is contracted to the point $z=a$.

We surround the point a by two concentric circles, O_1 and O_2, with centers at a, and with radii r_1 and r_2, respectively. We also enclose a by an arbitrary contour C of length l such that

$$2\pi r_2 \geqq l \geqq 2\pi r_1,$$

with the condition that the contour C lie between the two circles and that it contract in the limit to a point. The contour C, then, divides the area, σ, between the two circles into two annular parts, σ_1, and σ_2, adjacent to circles O_1 and O_2, respectively, so that $\sigma_1+\sigma_2=\sigma$.

We make the proof of this theorem depend on the following lemma.

LEMMA. *The circulation theorem,*

$$2i \int_\sigma \frac{\partial f}{\partial \bar{z}} d\sigma = \int_C f(z)dz, \tag{4}$$

is extensible to an annular domain bounded by two closed contours.

Let C_1 be a closed contour lying within the area bounded by the closed contour C_2. Let C_3 be a closed contour in the annular domain determined by C_1 and C_2 and not inclosing C_1. We assume that the circulation theorem (4) is valid for every such

contour C_3 in the annular domain. We now join a point P_2 of C_2 with a point P_1 of C_1 without intersecting the curve C_3. We then distort the contour C_3 till it coincides with C_1, C_2 and the curve joining the two points. When we take the flow around this distorted contour, possibly keeping the area to the left, the line integrals from P_2 to P_1 and from P_1 to P_2 just cancel each other. And this evidently establishes the lemma.

In passing we may mention that this lemma is a simple generalization of the usual extension of the Cauchy first law. Also the circulation theorem

$$(4') \qquad -2i\int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma = \int_C f(z)d\bar{z},$$

involving the Kasner mean derivative, may be extended in a similar way.

The application of the lemma to the area σ between O_1 and O_2 gives

$$(5) \qquad \int_{O_2} f(z)dz - \int_{O_1} f(z)dz = 2i\int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma,$$

where the line integrals are both taken counter-clockwise around the point a. But the paths of integration are circles; the line integrals in (5) therefore have the same limits.

Thus

$$(6) \qquad \lim_{r_2 \to 0} \int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma \equiv 0.$$

When we apply the lemma to the areas σ_2 and σ_1, taking the line integrals counter-clockwise around a, as before, we may write

$$(7) \qquad \int_{O_2} f(z)dz - \int_C f(z)dz = 2i\int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\,d\sigma,$$

and

$$(8) \qquad \int_C f(z)dz - \int_{O_1} f(z)dz = 2i\int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\,d\sigma.$$

If we subtract (8) from (7) we will find that

$$\int_{O_2} f(z)dz + \int_{O_1} f(z)dz - 2\int_C f(z)dz = 2i\left[\int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\, d\sigma - \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\, d\sigma\right]. \tag{9}$$

If $f(z)$ is analytic at $z=a$, $\partial f/\partial \bar{z}$ will vanish in the limit, so that the right member of (9) becomes zero. The theorem is thus verified for this case. Also if $f(z)$ is regular at $z=a$, then $\partial f/\partial \bar{z}$ is unique and bounded at $z=a$. Thus for $\sigma_{1,2}$, that is, for σ_1 or σ_2, we have, if M is the maximum value of $\left|\partial f/\partial \bar{z}\right|$ in $\sigma_{1,2}$,

$$\left| \int_{\sigma_{1,2}} \frac{\partial f}{\partial \bar{z}}\, d\sigma \right| \leqq \int_{\sigma_{1,2}} \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma \leqq M\sigma_{1,2},$$

which goes to zero with $\sigma_{1,2}$. Thus the right member of (9) is zero in the limit and the theorem is again validated.

If $f(z)$ is neither analytic nor regular at a, then (6) is satisfied by the ultimate cancellation of the elements of the integrand or because of the rapidity with which the area σ goes to zero. It would appear to be a very special function which would satisfy the first condition, while (6) is evidently valid when

$$\lim_{r_2 \to 0} \int_\sigma \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma = 0. \tag{10}$$

For such functions, then,

$$\left| \int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\, d\sigma - \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\, d\sigma \right| \leqq \int_\sigma \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma,$$

so that in the limit the left member of (9) vanishes. We have thus proved the theorem under the restriction imposed by (10).

As a simple example we might take $1/z+\bar{z}$ as $f(z)$ with the origin as the point a. Here $\partial f/\partial \bar{z}=1$, so that

$$\int_\sigma \frac{\partial f}{\partial \bar{z}}\, d\sigma = \int_\sigma d\sigma = \pi(r_2^2 - r_1^2),$$

which evidently goes to zero with r_2. We have

$$\int_{O_2^{\circlearrowright}} \left(\frac{1}{z} + \bar{z}\right) dz = 2\pi i + 2\pi i r_2^2,$$

which in the limit is $2\pi i$. Thus, by the theorem,

$$\lim_{C\to 0} \int_C \left(\frac{1}{z} + \bar{z}\right) dz = 2\pi i,$$

and this result can easily be verified. From analytic function theory $\int_C dz/z = 2\pi i$, while if we use r for the maximum value of $\bar{z}$, we have

$$\left| \int_C \bar{z} dz \right| \leqq \int_C |\bar{z}||dz| \leqq rl \leqq 2\pi r_2^2,$$

which goes to zero with r_2.

3. *Regular Functions.* We define a function to be regular in a domain D of the complex plane when it possesses a differential at every point of the domain. If $f(z)$ has a differential at every point $z = a$ of the domain, then

$$f(z) - f(a) \equiv \frac{\partial f}{\partial z}(z - a) + \frac{\partial f}{\partial \bar{z}}(\bar{z} - \bar{a}) + \eta, \tag{11}$$

where $\lim_{z\to a} \eta = 0$.

THEOREM 2. *The residue of every polygenic function regular in a domain D is identically zero at every point of the domain.*

In evaluating the integral in (1), we have

$$\begin{aligned} \int_0 f(z)dz &\equiv \int_0 f(a)dz + \int_0 [f(z) - f(a)]dz \\ &= f(a)\int_0 dz + \frac{\partial f}{\partial z}\int_0 (z - a)dz \\ &\quad + \frac{\partial f}{\partial \bar{z}}\int_0 (\bar{z} - \bar{a})dz + \int_0 \eta dz. \end{aligned}$$

The first two integrals in this last member vanish identically since the integrands are analytic functions, while each of the last two integrals contains a factor $r^{1+\epsilon}$, where $0 < \epsilon \leqq 1$; thus these integrals go to zero with r. Hence $R_z \equiv 0$.

In a similar way we can show that $R_{\bar{z}}$ is also zero. Hence $R \equiv 0$ at every point of the domain.

4. *Functions Defined by a Laurent Series.* In this section we shall be interested in the residues of a class of polygenic functions defined by a Laurent series. It will be sufficiently general to study the series in the neighborhood of the origin. Thus let

$$(12) \qquad f(z) \equiv \sum_{n=1}^{m} \sum_{k=0}^{n} \frac{a_{n-k,k}}{z^{n-k}\bar{z}^{k}} + a_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} b_{n-k,k} z^{n-k}\bar{z}^{k};$$

we proceed to find the residue R_z of $f(z)$ according to definition (1). The contribution to R_z due to the term whose denominator is $z^{n-k}\bar{z}^{-k}$, $(n \neq 1)$, will contain the factor

$$\int_0 \frac{dz}{z^{n-k}\bar{z}^{k}}.$$

For z we write $z = re^{i\theta}$; then $\bar{z} = re^{-i\theta}$ and $dz = rie^{i\theta}\, d\theta$, so that we have

$$\int_0 \frac{dz}{z^{n-k}\bar{z}^{k}} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{r^{n}[i(n-2k)\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1+2k-n)\theta}d\theta \equiv 0,$$

since $1+2k-n$ is a positive or negative integer. Similarly the contribution to R_z by the term $b_{n-k,k}\, z^{n-k}\bar{z}^{k}$ will contain the factor $\int_0 z^{n-k}\bar{z}^{k}dz$. Hence

$$\int_0 z^{n-k}\bar{z}^{k}dz = \int_0 ir^{n+1}e^{i(n-2k+1)}d\theta = ir^{n+1}\cdot 0 = 0.$$

Since the residue of a_0 is zero, the only contribution to R_z will come from the term involving $a_{1,0}$ as a factor. Therefore

$$R_z = \lim_{r\to 0} \frac{a_{1,0}}{2\pi i} \int_0 \frac{dz}{z} = \frac{a_{1,0}}{2\pi i} \int_0^{2\pi} id\theta = \frac{a_{1,0}}{2\pi i} \cdot 2\pi i = a_{1,0}.$$

The same procedure will convince one that the only contribution to $R_{\bar{z}}$ will be furnished by the term involving the factor $a_{0,1}$. Then by definition

$$R_{\bar{z}} = \lim_{r\to 0} \frac{-1}{2\pi i} \int_0 \frac{a_{0,1}d\bar{z}}{\bar{z}} = \lim_{r\to 0} \frac{ia_{0,1}}{2\pi i} \int_0^{2\pi} d\theta = \frac{2\pi i a_{0,1}}{2\pi i} = a_{0,1}.$$

Hence the total residue at the origin of a function defined by a Laurent series is

$$R = R_s + R_3 = a_{1,0} + a_{2,1}.$$

In conclusion it should be stated that R_3 is the negative of $\overline{R}$ defined in the article in the Transactions (loc. cit.); this change is also carried over into the definition for the total residue. The reason for this change is partly evident in the results just obtained; then this change of sign brings R_3 into accord with the mean derivative and the circulation theorem (4′). Also a slight change in the proof of Theorem 1 for R_s will establish the theorem for R_3.

THE UNIVERSITY OF MICHIGAN

NOTE ON A MERSENNE NUMBER

BY R. E. POWERS

I have recently determined by the computation of Lucas' series 4, 14, 194, · · · *that the number $N = 2^{241} - 1$ is composite, since the 240th term of the series is congruent to

$$-\ 98\ 6778335538\ 8807227981\ 3604528486\ 9326522489\ 7467133466$$
$$0099172867\ 1619979800 \pmod{N}.$$

This term would be zero if N were prime.

The square of each term was obtained by means of a computing machine, D. N. Lehmer's *cross-multiplication*† being used; and these squares, diminished by 2, were divided by N by hand, with the aid of a table of the 1000 multiples of N: N, $2N$, $3N$, · · · , $1000N$, the quotients being thus obtained three or more digits at a time, and the computation was checked throughout by the four moduli 9, 10^3+1, 10^4+1, and 10^7+1.

DENVER, COLORADO

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* This Bulletin, vol. 38 (1932), p. 383.

† American Mathematical Monthly, vol. 30 (1923), p. 67, and vol. 33 (1926), p. 199.

CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH CERTAIN SYMMETRIC EQUATIONS IN n RECIPROCALS*

BY NORMA STELFORD AND H. A. SIMMONS

1. *Introduction.* In an article dealing with this subject Simmons† stated without proof two general theorems whose proofs are to be obtained by making certain modifications in the theory of his article, which will be referred to in the sequel as I (for paper I). These theorems will be stated after a few definitions from I are recalled.

Kellogg solution. If a solution $x \equiv (x_1, x_2, \cdots, x_n)$ of any given symmetric equation in n reciprocals is obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ (all positive integers) in this order, one at a time, we shall denote it by w and call it the Kellogg solution of the given equation. Thus $x \equiv (2, 3, 6)$ is the Kellogg solution w of the equation $x_1^{-1}+x_2^{-1}+x_3^{-1}=1$.

E-solution. A solution $x \equiv (x_1, x_2, \cdots, x_n)$ of a given symmetric equation in n reciprocals is called an E-solution if $x_1, x_2, \cdots, x_{n-1}$ are positive integers and $x_1 \leqq x_2 \leqq \cdots \leqq x_n$.

Polynomial $P(x)$. Let $P(x_1, x_2, \cdots, x_n) \equiv P(x)$ be any polynomial which is symmetric in the n variables x_i, contains one or more positive coefficients and no negative coefficient, and is not identically a constant.

$\Sigma_{i,j}(x)$. With $i \geqq 0$ and j equal to integers, we let $\Sigma_{i,j}(x)$ stand for the jth elementary symmetric function of the i variables $x_1, x_2, \cdots, x_i$; with the customary understanding that

$$\Sigma_{i,j}(x) \begin{cases} \equiv 0 \text{ when } i < j \text{ and also when } j < 0, \\ \equiv 1 \text{ when } j = 0. \end{cases}$$

We now state the two theorems referred to above with the numbering of I.

THEOREM 4. *If in the equation*

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* Presented to the Society, April 6, 1934.

† See H. A. Simmons, Transactions of this Society, vol. 34 (1932), pp. 876–907.

$$\Sigma_{n,r}(1/x) + \lambda_{r+1}\Sigma_{n,r+1}(1/x) + \lambda_{r+2}\Sigma_{n,r+2}(1/x) + \cdots + \lambda_s\Sigma_{n,s}(1/x) = b/a, \qquad a \equiv [(c+1)b - 1], \tag{21}$$

where b and c are positive integers, either $\lambda_p = 1$, $(p = r+1, \cdots, s)$, or λ_{r+1} is an integer $\geqq 0$ and $\lambda_p = 0$, $(p = r+2, \cdots, s)$, the largest number that exists in any E-solution of the resulting equation (21) *is the w_n of the corresponding solution w defined by the recurrence relations*

$$\begin{aligned} w_p &= 1, \qquad (p = 1, \cdots, r-1), \qquad w_r = c+1, \\ w_{p+1} &= a[\Sigma_{p,p-r+1}(w) + \lambda_{r+1}\Sigma_{p,p-r}(w) + \lambda_{r+2}\Sigma_{p,p-r-1}(w) \\ &\quad + \cdots + \lambda_s\Sigma_{p,p-s+1}(w)] + 1, \quad (p = r, \cdots, n-2), \\ w_n &= a[\Sigma_{n-1,n-r}(w) + \lambda_{r+1}\Sigma_{n-1,n-r-1}(w) + \lambda_{r+2}\Sigma_{n-1,n-r-2}(w) \\ &\quad + \cdots + \lambda_s\Sigma_{n-1,n-s}(w)]. \end{aligned} \tag{23}$$

Furthermore, in each of these cases, w_n appears in but one E-solution of the equation (21) *in question.*

THEOREM 5. *In each of the two cases of Theorem* 4, *if X is an E-solution of equation* (21) *and is different from the w of that equation, then $P(X) < P(w)$.*

Here we shall extend all theorems of I that relate to maximum numbers by considering cases in which the right members, b/a, of the equations in question are much more general than they were in I. For (a very simple) example, consider

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{3}{3c-2},$$

where $c \geqq 3$ is odd. The Kellogg solution of this equation is readily found to be $x = w$, where $w_1 = c$, $w_2 = 2^{-1}(3c^2 - 2c + 1)$, $w_3 = (3c-2)w_1w_2$. We show later that the theorems above hold for this solution w. This conclusion does not follow from I because $(3/(3c-2)) - (1/w_1)$ is not a unit fraction, while in I, $(b/a) - (1/(w_1 \cdots w_r))$, $(r = 1, \cdots, n-1)$, was required to be a unit fraction. Our new theorems are stated in §§6, 9, 10.

We divide the portion of this paper from §2 to the end into two parts. In Part 1, §§2 to 5, we prove a case of Theorem 4; in Part 2, we develop our extensions as far as seems desirable.

Except where the contrary is stated, we employ here the definitions and notation of I. The analogs of equation (*i*) and

Theorem i of I, if written here, are denoted by (ia) and Theorem ia, respectively, and proofs that are to be obtained as were analogous proofs of I are omitted. Consequently, in Part 1 we only state our modifications of certain definitions and lemmas and prove inequality (46a).

Part 1. Proof of one Case of Theorem 4

2. *New Relations.* We shall prove the part of Theorem 4, §23 of I, for which $\lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_s=1$, that is, the case where equation (21) is

$$\phi_n(1/x) = b/a, \tag{21a}$$

where $\phi_n(1/x)\equiv\Sigma_{n,r}(1/x)+\Sigma_{n,r+1}(1/x)+\cdots+\Sigma_{n,s}(1/x)$, and r, s, n are positive integers such that $r<s\leqq n$.

If in the last displayed equation n is replaced by p, $r\leqq p\leqq n$, our definition of $\phi_p(1/x)$ is obtained.

The analogs here of (26), (28), (29), (30), (32) of I are as follows.

$$\begin{aligned} w_p &= 1, \qquad (p = 1, \cdots, r-1), \qquad w_r = c+1, \\ w_{p+1} &= a\left[\Sigma_{p,p-r+1}(w) + \Sigma_{p,p-r}(w) + \cdots\right. \\ &\qquad \left. + \Sigma_{p,p-s+1}(w)\right] + 1, \quad (p = r, \cdots, n-2), \\ w_n &= a\left[\Sigma_{n-1,n-r}(w) + \Sigma_{n-1,n-r-1}(w) + \cdots\right. \\ &\qquad \left. + \Sigma_{n-1,n-s}(w)\right]. \end{aligned} \tag{26a}$$

$$\phi_p(1/w) = (bw_1\cdots w_p - 1)/(aw_1\cdots w_p), \qquad (p = r, \cdots, n-1; \text{ see (27) of I}). \tag{28a}$$

$$\phi_p(1/X) \leqq (bX_1\cdots X_p - 1)/(aX_1\cdots X_p), \qquad (p = r, \cdots, n-1), \tag{29a}$$

which is to be proved as was (29) of I.

$$\phi_{n-1}(1/X) \leqq \phi_{n-1}(1/w), \qquad (1 \leqq r < s \leqq n). \tag{30a}$$

$$\begin{aligned} &\Sigma_{n-1,r-1}(1/X) + \Sigma_{n-1,r}(1/X) + \cdots + \Sigma_{n-1,s-1}(1/X) \\ &\quad < \Sigma_{n-1,r-1}(1/w) + \Sigma_{n-1,r}(1/w) + \cdots + \Sigma_{n-1,s-1}(1/w), \\ &\qquad\qquad (1 \leqq r < s \leqq n). \end{aligned} \tag{32a}$$

3. *New Definitions of Sets σ, τ, and of the Transformation. Sets** σ, τ.* Let λ be a fixed positive integer such that $r \leqq \lambda \leqq n$, where r and n are as they are defined for (21a). We shall call $x_1 \ldots_\lambda$ a set σ (relative to the w of (26a)) if, and only if, $\phi_p(1/x) \leqq \phi_p(1/w)$ for every positive integer p such that $r \leqq p \leqq \lambda$. We shall call $x_1 \ldots_{(\lambda+1)}$ a set τ if, and only if, with λ a positive integer such that $r \leqq \lambda \leqq n-2$, $x_1 \ldots_{(\lambda+1)}$ is not, and $x_1 \ldots_\lambda$ is, a set σ.

The new transformation.† Suppose $X_1' \ldots_\nu$ contains at least one element of each of the classes A and B defined in I. Then we define our transformation of $X_1 \ldots_\nu$ into a new set $X_1' \ldots_\nu$ by (t_{3a}) or (t_{4a}):

$$(33a)\qquad \begin{aligned} (t_{3a}) \qquad & X_p' = X_p (p \neq q_1, q, p \leqq \nu),\ X_{q_1}' = w_{q_1}, \\ & \qquad \phi_\nu(1/X') = \phi_\nu(1/X); \\ (t_{4a}) \qquad & X_p' = X_p (p \neq q_1, q, p \leqq \nu),\ X_{1q}' = w_{1q}, \\ & \qquad \phi_\nu(1/X') = \phi_\nu(1/X), \end{aligned}$$

according as (t_{3a}) requires X_{iq}' to be not greater than w_{iq} or greater than w_{1q}, respectively.

4. *Analogs of Lemmas* 6, 7, 8.

LEMMA 6a. (i) *If $X_1 \ldots_k$ is a set τ, $X_1 \ldots_k$ is transformable.*‡ (ii) *If $X_1 \ldots_k$ is a set τ or a transformable set σ for which $r < k \leqq n$, and if t is a positive integer, application of* (33a) *with $\nu = k$ to $X_1 \ldots_k$ yields a set $X_1' \ldots_k$ such that* (34a), *identical with* (34), *is true, and*

$$(35a)\qquad \phi_p(1/X') \leqq \phi_p(1/w), \text{ for } p = r, \cdots, {}_1q - 1;$$

[(36a) is not needed];

$$(37a)\qquad \phi_p(1/X') < \phi_p(1/X), \text{ for } p = {}_1q, \cdots, k-1;$$

$$(38a)\qquad \phi_p(1/X') = \phi_p(1/X), \text{ for } p = k.$$

LEMMA 7a. *If s_1, s_2, λ are integers $\geqq 0$ and $s_1 \geqq s_2, \lambda > 0$, then*

$$(1^{s_1})(1^{s_2}) > (1^{s_1+\lambda})(1^{s_2-\lambda}).$$

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* See pp. 889–890 of I.
† See p. 891 of I.
‡ See p. 891 of I.

In proving the next lemma by our method, Lemma 7a would be used.

LEMMA 8a. *If u, v, γ, t are integers with $u>v\geqq\gamma\geqq 1$, $0\leqq t\leqq(\gamma-1)$, and $x_i>0$ for $i=1,\cdots,u$, then*

$$\Sigma_{u,\gamma}(1/x)\Sigma_{v,t}(1/x) > \Sigma_{u,t}(1/x)\Sigma_{v,\gamma}(1/x).$$

5. *Analog of Relation* (46) *of* I. From (37a) and (38a) one may obtain (46a) below as (46) was obtained from (37) and (38) of I:

$$\begin{aligned}(46a)^*\quad &[\Sigma'_{k-2,r-1}+\Sigma'_{k-2,r}+\cdots+\Sigma'_{k-2,s-1}][\Sigma'_{p-2,r-2}\\ &+\Sigma'_{p-2,r-1}+\cdots+\Sigma'_{p-2,s-2}]\\ &>[\Sigma'_{k-2,r-2}+\Sigma'_{k-2,r-1}+\cdots+\Sigma'_{k-2,s-2}][\Sigma'_{p-2,r-1}\\ &+\Sigma'_{p-2,r}+\cdots+\Sigma'_{p-2,s-1}].\end{aligned}$$

With

$$c_1\equiv\Sigma'_{k-2,r-1}+\Sigma'_{k-2,r}+\cdots+\Sigma'_{k-2,s-2},$$
$$c_2\equiv\Sigma'_{p-2,r-1}+\Sigma'_{p-2,r}+\cdots+\Sigma'_{p-2,s-2},$$

(46a) may be written

$$(\Sigma'_{k-2,s-1}+c_1)(\Sigma'_{p-2,r-2}+c_2) > (\Sigma'_{k-2,r-2}+c_1)(\Sigma'_{p-2,s-1}+c_2).$$

PROOF OF (46a). That this inequality is true follows from the inequalities I_1, I_2, I_3 below, which we presently prove. In arriving at (46a), one finds that it is to hold under the hypotheses $1\leqq r\leqq {}_1q\leqq p<k$, with ${}_1q\geqq 2$. These relations are understood to hold in this proof.

$$(I_1)\qquad c_1\Sigma'_{p-2,r-2} > c_2\Sigma'_{k-2,r-2},$$

$$(I_2)\qquad c_2\Sigma'_{k-2,s-1} \geqq c_1\Sigma'_{p-2,s-1},$$

$$(I_3)\qquad \Sigma'_{k-2,s-1}\Sigma'_{p-2,r-2} \geqq \Sigma'_{k-2,r-2}\Sigma'_{p-2,s-1}.$$

The terms in c_1 are $\Sigma'_{k-2,\lambda}$, $(\lambda=r-1,\cdots,s-2)$; those in c_2, $\Sigma'_{p-2,\lambda}$. Hence (I_1), (I_2) may be written

$$(I_1')\qquad \Sigma'_{k-2,\lambda}\Sigma'_{p-2,r-2} > \Sigma'_{k-2,r-2}\Sigma'_{p-2,\lambda},$$

$$(I_2')\qquad \Sigma'_{k-2,s-1}\Sigma'_{p-2,\lambda} \geqq \Sigma'_{k-2,\lambda}\Sigma'_{p-2,s-1},$$

* $\Sigma'_{i,j}\equiv\Sigma'_{i,j}(1/X)$.

respectively. If $p-2 \geqq \lambda$, (I_1') holds by Lemma 8a with $u=k-2$, $v=p-2$, $\gamma=\lambda$, and $t=r-2$. If $(p-2)<\lambda$, then since $p \geqq r$ the left side of (I_1') is positive and the right side is zero, so that (I_1') holds. Consider (I_2'). If $(p-2) \geqq (s-1)$, (I_2') holds with $>$ by Lemma 8a, with $u=k-2$, $v=p-2$, $\gamma=s-1$, $t=\lambda$. If $(p-2)$ $<(s-1)$, the right side of (I_2') is zero and the left side is surely not negative. Therefore (I_2') holds with $\geqq$. The proof for (I_2) is a special case of that for (I_2').

Part 2. Extensions of Previous Theory

6. *A New Theorem on Equations of the Form* $\Sigma_{n,1}(1/x)=b/a$. The article I contains results for equations of this type only when $a=[(c+1)b-1]$ and b, c are positive integers. This case is covered by taking $\mu=1$ in the new Theorem A below.

Theorem A. *Suppose* a, b *and* μ, $\mu \leqq (n-1)$, *are given positive integers, with* a *and* b, $b \leqq a$, *relatively prime, and that there exists a set of* n *numbers* $w \equiv (w_1, w_2, \cdots, w_n)$ *with the following properties*:

(1°) *it is an E-solution of the equation* $\Sigma_{n,1}(1/x)=b/a$;

$$(2°) \qquad \frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=\frac{bw_1\cdots w_p-1}{aw_1\cdots w_p}^{*}$$

for every positive integral value of p *for which* $\mu \leqq p \leqq (n-1)$ *and for no smaller value of* p;

(3°) *if* $x=X$, *where* $X \neq w$, *is an E-solution of the equation in* (1°), *then for every positive integral value of* $p \leqq \mu$ *for which* $X_{1\cdots p} \neq w_{1\cdots p}$, *the inequality* $\Sigma_{p,1}(1/X)<\Sigma_{p,1}(1/w)$ *holds.*

For such a set w *the following conclusions hold*:

(i) w_n *is the largest number that exists in any E-solution of the equation in* (1°) *and* w_n *appears in no E-solution of this equation except* w;

(ii) $P(x)<P(w)$.†

Proof of (i). It follows from hypotheses $(j°)$, $(j=1, 2, 3)$, that the w described in Theorem A is the Kellogg solution of the equation in (1°). This fact will be used in the rest of the proof of (i), which we presently make by considering separately the following cases: (α) $\mu=n-1$; (β) $\mu=n-2$; (γ) $\mu \leqq (n-3)$.

* See equation (28) of I, with $r=1$.

† See p. 887 of I.

(α) In this trivial case, (3°) requires that conclusion (i) be true.

(β) If $X_1 \dots_\mu = w_1 \dots_\mu$, since $X \neq w$ is an E-solution of the equation in (1°) while w is its Kellogg solution, $X_{\mu+1} \equiv X_{n-1} > w_{n-1}$, and (i) is true. If $X_1 \dots_\mu \neq w_1 \dots_\mu$, we reach the desired conclusion as follows. From the hypotheses (j°), ($j = 1, 2, 3$), and the assumption that $X_1 \dots_\mu \neq w_1 \dots_\mu$, with $\mu = n-2$, we have

$$\Sigma_{p,1}(1/X) \leqq \Sigma_{p,1}(1/w), \qquad (p = 1, \cdots, \mu - 1);$$

$$(1) \qquad \Sigma_{n-2,1}(1/X) < \Sigma_{n-2,1}(1/w) \equiv \frac{bw_1 \cdots w_{n-2} - 1}{aw_1 \cdots w_{n-2}};$$

$$(2) \qquad \Sigma_{n-1,1}(1/X) \leqq \frac{bX_1 \cdots X_{n-1} - 1}{aX_1 \cdots X_{n-1}}, \qquad \text{[see (29) of I]};$$

and we wish to contradict the assumption which we now make, namely that

$$(3) \qquad \Sigma_{n-1,1}(1/X) \geqq \Sigma_{n-1,1}(1/w) \equiv \frac{bw_1 \cdots w_{n-1} - 1}{aw_1 \cdots w_{n-1}}.$$

Relations (2) and (3) imply that

$$(4) \qquad X_1 \cdots X_{n-1} \geqq w_1 \cdots w_{n-1};$$

(1) and (3), that $X_1 \dots_{(n-1)}$ is transformable. Since X is an E-solution, it follows now that when exhaustive applications of (33a) for $X_1 \dots_{(n-1)}$ are made, the following relations hold [see (53) of I]:

$$(5) \qquad X_1 \cdots X_{n-1} < X_1' \cdots X_{n-1}' \leqq w_1 \cdots w_{n-1}.$$

Since (4) and (5) are contradictory, we conclude that (3) is false. Consequently conclusion (i) holds in case (β).

(γ) The demonstration that is required here is essentially the same as that given under (β) for the case $X_1 \dots_\mu \neq w_1 \dots_\mu$; induction integers k, $(k+1)$, where $\mu \leqq k \leqq (n-2)$, here take the place of $(n-2)$, $(n-1)$, respectively, there. Hence conclusion (i) holds in case (γ).

PROOF OF (ii). This result can be reached by the method that was employed in proving Theorem 3 of I, since inequalities analogous to those in (53) of I hold here. Hence Theorem A is true.

7. *An Example of Takenouchi.* The Takenouchi equation*

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=\frac{5}{11} \tag{6}$$

has, as we presently prove, no solution w of the type hypothesized in Theorem A. For the Kellogg solution of (6) is $X=w=(3, 9, 99)$, and it is easily seen that for this X there does not exist an integer μ of the type specified in (2°). One may also negate the existence here of the w of Theorem A by observing with Takenouchi that the set (4, 5, 220) satisfies (6), while $(4^{-1}+5^{-1})>(3^{-1}+9^{-1})$, so that $(3, 9)=(w_1, w_2)$ does not accord with hypothesis (3°).

8. *Corollaries of Theorem* A. We have seen in I that when $a=[(c+1)b-1]$, the w of Theorem A always exists (case $\mu=1$). When $a\neq[(c+1)b-1]$ there are, as we point out in the corollaries below, numerous cases for which the w of Theorem A exists with $\mu=2$. In cases where $\mu>2$, we have no better method than that of trial and error for testing a given E-solution X to see whether it satisfies hypothesis (3°). However, when $\mu=2$ the following Auxiliary Theorem shows that hypothesis (3°) is always satisfied when (1°) and (2°) are. For Theorem A only the case $r=1$, $\lambda_{r+1}=\lambda_2=0$ of this Auxiliary Theorem is needed; other cases of it are similarly useful in connection with Theorem *i*a, $(i=2, 3, 4, 5)$, of I.

AUXILIARY THEOREM. *Let X be an E-solution of the equation*

$$\Sigma_{n,r}(1/x)+\lambda_{r+1}\Sigma_{n,r+1}(1/x)+\cdots+\lambda_s\Sigma_{n,s}(1/x)=b/a,$$

where $X\neq w$, w being the Kellogg solution of this equation; the λ_i, $(i=r+1, \cdots, s, s>r)$, are integers $\geqq 0$; $n\geqq r+2$;† and a, b are as in Theorem A *except that we now suppose that*

$$\frac{1}{w_1\cdots w_r}\neq\frac{bw_1\cdots w_r-1}{aw_1\cdots w_r};$$

and that

* See p. 92 of loc. cit. in footnote 3, p. 876, of I.

† This restriction insures the presence of at least two terms free of x_n in the left member of the equation just displayed. When $n=r+1$, Theorem A has no content beyond the case $\mu=1$, which was handled in I.

$$\frac{1}{w_1 \cdots w_r} + \frac{1}{w_{r+1}}\left[\Sigma_{r,r-1}(1/w) + \frac{\lambda_{r+1}}{w_1 \cdots w_r}\right] = \frac{bw_1 \cdots w_{r+1} - 1}{aw_1 \cdots w_{r+1}}.$$

Then we conclude that* $(X_1 \cdots {}_{(r+1)}$ *is a set* σ *in the sense that*)

$$\frac{1}{X_1 \cdots X_r} \leqq \frac{1}{w_1 \cdots w_r};$$

$$\frac{1}{X_1 \cdots X_r} + \frac{1}{X_{r+1}}\left[\Sigma_{r,r-1}\frac{1}{X} + \frac{\lambda_{r+1}}{X_1 \cdots X_r}\right] < \frac{bw_1 \cdots w_{r+1} - 1}{aw_1 \cdots w_{r+1}}.$$

The proof of this theorem can be made by reasoning of the type that was used on p. 897 of I. Each of the three corollaries below is a case $\mu = 2$ of Theorem A.

COROLLARY 1. *For the equation* $\Sigma_{n,1}(1/x) = 5/17$, *with* $n > 2$, *the* w *of Theorem* A *is* $w_1 = 4$, $w_2 = 23$, $w_{i+1} = 17w_1 \cdots w_i + 1$, $(i = 2, \cdots, n-2)$, $w_n = 17w_1 \cdots w_{n-1}$.

COROLLARY 2. *For the equation* $\Sigma_{n,1}(1/x) = 3/(3c-2)$, *with* $n > 2$, *if* $c \geqq 3$ *is odd, the* w *of Theorem* A *is defined by the equations* $w_1 = c$, $w_2 = 2^{-1}(3c^2 - 2c + 1)$, $w_{i+1} = (3c-2)w_1 \cdots w_i + 1$, $(i = 2, \cdots, n-2)$, $w_n = (3c-2)w_1 \cdots w_{n-1}$.

COROLLARY 3. *For the equation* $\Sigma_{n,1}(1/x) = b/a$, *with* $n > 2$ *and* $a \equiv [(c+1)b - 2]$, *if* $b > 1$ *is odd and* c *is even, the set* w *of Theorem* A *is defined by* $w_1 = c+1$, $w_2 = 2^{-1}[a(c+1)+1]$, $w_{i+1} = aw_1 \cdots w_i + 1$, $(i = 2, \cdots, n-2)$, $w_n = aw_1 \cdots w_{n-1}$.

9. *Generalizations of Theorem* 2, 3 *of* I.

THEOREM 2a. *Suppose* a, b *and* μ, *where* $r \leqq \mu \leqq (n-1)$, *are given positive integers, with* a *and* b, $b \leqq a$, *relatively prime, and that there exists a set of* n *numbers* $w \equiv (w_1, \cdots, w_n)$ *with the following properties*:

(1°) *it is an E-solution of the equation*

$$\Sigma_{n,r}(1/x) = b/a, \qquad (1 < r < (n-1));\dagger$$

* If the sign $<$ in the second relation below were replaced by the sign $\leqq$, $X_1 \cdots {}_{(r+1)}$ would still be a set σ in a sense perfectly analogous to that in which the term *set* σ was used in I.

† The case $r = 1$ has been treated in Theorem A; the case $r = (n-1)$, in I. Hence only the cases $1 < r < (n-1)$ remain for consideration.

$$(2^\circ) \qquad \Sigma_{p,r}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p}$$

for every positive integral value of p for which $\mu \leqq p \leqq (n-1)$ and for no positive integral value of p less than μ;
(3°) if $x = X$, where $X \neq w$, is an E-solution of the equation in (1°), then for every positive integral value of p such that $r \leqq p \leqq \mu$ for which $X_1 \ldots_p \neq w_1 \ldots_p$, the relation $\Sigma_{p,r}(1/X) \leqq \Sigma_{p,r}(1/w)$ holds.

Then w_n is the largest number that exists in any E-solution of the equation in (1°) and w_n appears in no E-solution of this equation except w.

THEOREM 3a. *If $x \neq w$ is any E-solution of the equation considered in Theorem* 2a, $P(x) < P(w)$. [See Theorem 3 of I.]

Since the methods which we have used in proving Theorem A together with those of I suffice to prove the last two theorems we omit their proofs.

The following corollary shows that Theorem *i*a, $(i = 2, 3)$, has content for a case in which $\mu > r$.

COROLLARY 4. *The Kellogg solution of the equation* $\Sigma_{n,3}(1/x) = 5/11$, $n \geqq 5$, *is* $x = w$, *where* $w_1 = w_2 = 1$, $w_3 = 3$, $w_4 = 14$, $w_{i+1} = 11\Sigma_{i,i-r+1}(w) + 1$, $(i = 4, \cdots, n-2)$, $w_n = 11\Sigma_{n-1,n-r}(w)$.

It is easy to show that in this example the μ of Theorem 2a exists and equals 4.

10. *Generalizations of Theorems* 4, 5.

THEOREM 4a. *Suppose a, b, μ are as in Theorem* 2a *and that there exists a set of n numbers $w > (w_1, \cdots, w_n)$ with the following properties:*
(1°) it is an E-solution of the equation $\Sigma_{n,r}(1/x) + \lambda_{r+1}(1/x) = b/a$, in which λ_{r+1} is an integer $\geqq 0$;

$$(2^\circ) \qquad \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p}$$

for every positive integral value of p for which $\mu \leqq p < n$ (and for no positive integral value of p less than μ when $\mu > r$);
(3°) if $x = X$, where $X \neq w$, is an E-solution of the equation in (1°), then for every positive integral value of p such that $r \leqq p \leqq \mu$ for which $X_1 \ldots_p \neq w_1 \ldots_p$, the relation

$$\Sigma_{n,r}(1/X) + \lambda_{r+1}\Sigma_{p,r+1}(1/X) \leqq \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w)$$

holds.

In such a set w, w_n is the largest number that exists in any E-solution of the equation in (1°) *and w_n appears in no E-solution of this equation except w. Furthermore a similar statement holds when the left member of the equation in* (1°) *is replaced by*

$$\Sigma_{n,r}(1/x) + \Sigma_{n,r+1}(1/x) + \cdots + \Sigma_{n,s}(1/x),$$

where, as heretofore, s is a positive integer and $r<s\leqq n$.

THEOREM 5a. *In each of the two cases of Theorem* 4a, *if X is an E-solution of the given equation and $\neq w$, the Kellogg solution of that equation, then $P(X)<P(w)$.*

The following corollaries show that the theorems of this section have content in cases where $\mu=2$ when $r=1$.

COROLLARY 5. *For the equation $\Sigma_{n,1}(1/x)+3\Sigma_{n,2}(1/x)=5/17$, with $n>2$, the set w of Theorem* 4a *is given by $w_1=4$, $w_2=40$, $w_{i+1}=17[\Sigma_{i,i}(w)+3\Sigma_{i,i-1}(w)]+1$, $(i=2,\cdots,n-2)$, and $w_n=17[\Sigma_{n-1,n-1}(w)+3\Sigma_{n-1,n-2}(w)]$.*

COROLLARY 6. *For the equation $\Sigma_{n,1}(1/x)+\Sigma_{n,2}(1/x)=4/13$, with $n>2$, the w of Theorem* 4a (*see last sentence of that theorem*) *is given by $w_1=4$, $w_2=22$, $w_{i+1}=13[\Sigma_{i,i}(w)+\Sigma_{i,i-1}(w)]+1$, $(i=2,\cdots,n-2)$, and $w_n=13[\Sigma_{n-1,n-1}(w)+\Sigma_{n-1,n-2}(w)]$.*

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ERRATA

The following changes should be made in the present volume (Vol. 40) of this Bulletin:

Page 93, last line of Theorem 2, insert before the words "*is that*" the words "*and that $f_m(x)$ be continuous.*"

Pages 413–416, change f to f_0 in the following places: in the statement of Theorem 2 on p. 413; in the statement of Theorem 6 on p. 415; and in five places occurring in the first six lines of p. 416.

IMPLICIT FUNCTIONS OF ALMOST PERIODIC FUNCTIONS*

BY R. H. CAMERON†

1. *Introduction.* It is a well known fact that a uniformly continuous single-valued function of a uniformly almost periodic (u.a.p.) function is u.a.p., and that its module is contained in the module of the original function. However, the same statements cannot be made concerning multiple-valued functions; for instance, a u.a.p. function can be constructed which is never zero and whose two continuous square root functions are neither of them u.a.p. (or even Stepanoff a.p.). Moreover, when a square root (or other multiple-valued) function is a.p., it need not have a module contained in the module of the original. This is illustrated by e^{ix} whose module consists of all integers, but whose square roots $e^{ix/2}$ and $-e^{ix/2}$ have modules consisting of all integers and half integers.

It is the aim of this paper to obtain sufficient conditions under which multiple-valued functions (implicit functions) of an a.p. function are a.p., and to investigate the module of such solutions. In the main, u.a.p. functions will be considered, but some results will also be obtained for S.a.p. functions.

2. *Statement of Theorem* 1. In the following theorem, x denotes the real variable while $f(x)$, z, and $F(x, z)$ are either all real or all complex.

THEOREM 1. *Let $f(x)$ be a continuous bounded function defined for all values of x; let $\mathfrak{C}$ be the set of values which $f(x)$ can assume and let $\mathfrak{S}$ be the set of ordered pairs (x, z) such that $|f(x)-z|<\mu$, $\mu>0$. Let $F(x, z)$ be a function having the following properties:*

(a) *$F(x, z)$ is u.a.p. in x uniformly in z for all z in $\mathfrak{C}$.*

(b) *$F(x, z)$ is uniformly continuous in z uniformly with respect to x for all (x, z) on $\mathfrak{S}$.*

(c) *$\partial F/\partial z = F_z(x, z)$ exists everywhere on $\mathfrak{S}$.*

(d) *$F[x, f(x)]=0$ and $|F_z[x, f(x)]|>\lambda>0$ for all x.*

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(e) *$F_x(x, z)$ is continuous in z at $z=f(x)$ uniformly for all x. Finally, let $\mathfrak{R}$ be the set of sequences of real numbers $h_1, h_2, \cdots$ such that $\lim_{k\to\infty} f(h_k)$ exists and $\lim_{k\to\infty} F(x+h_k, z) = F(x, z)$ uniformly for all x and all z in $\mathfrak{C}$.*

Then it follows that

(a) *$f(x)$ is u.a.p.*

(b) *The number P of numbers which can be approached by sequences of $\mathfrak{R}$ is finite.*

(c) *The module of $f(x)$ is contained in the set of numbers obtained by dividing by P each element of the common module of $F(x, z)$ for z in $\mathfrak{C}$.*

3. *Proof.* Because of the hypotheses on $F(x, z)$, we have a uniform implicit function theorem as follows. There exist positive numbers η and ν such that to each pair (x, w) of which w is a complex number satisfying $|w| \leqq \nu$ there corresponds one and only one complex number $Z_x(w)$ which satisfies simultaneously

$$|Z_x(w) - f(x)| \leqq \eta \text{ and } F[x, Z_x(w)] = w. \tag{1}$$

Moreover $Z_x(w)$ is uniformly continuous in w uniformly with respect to x for $|w| \leqq \nu$ and all x; and of course $Z_x(0) = f(x)$.

It will be shown in this section that if $h_1, h_2, \cdots$ belong to $\mathfrak{R}$, then $\lim_{k\to\infty} f(x+h_k)$ exists uniformly in x for all x. Let $h_1, h_2, \cdots$ be any sequence of $\mathfrak{R}$, and let $\epsilon > 0$. Then corresponding to ϵ_1, the smaller of ϵ and $\eta/2$, choose a positive number $\delta \leqq \nu$ such that whenever $|w| \leqq \delta$, then, for all x,

$$|Z_x(w) - f(x)| \leqq \epsilon_1. \tag{2}$$

Let N be so great that $|f(h_m) - f(h_n)| \leqq \epsilon_1$ when $m > N$ and $n > N$, and that, for all x and all z in $\mathfrak{C}$,

$$|F(x + h_m, z) - F(x + h_n, z)| \leqq \delta. \tag{3}$$

Now suppose that there exists for some $m > N$ and $n > N$ a real number x_0 such that

$$\eta > |f(x_0 + h_m) - f(x_0 + h_n)| > \epsilon_1; \tag{4}$$

and let

$$w_0 = F[x_0 + h_n, f(x_0 + h_m)]. \tag{5}$$

Then, since $F[x_0+h_m, f(x_0+h_m)]=0$, we have from (3) that $|w_0| \leqq \delta \leqq \nu$. It follows from the uniqueness of the function $Z_x(w)$ and from (1), (4), (5), that $Z_{x_0+h_n}(w_0)=f(x_0+h_m)$. Hence we deduce from (2) that $|f(x_0+h_m)-f(x_0+h_n)| \leqq \epsilon_1$, which contradicts (4) and shows that our supposition that (4) holds is never true. But the continuous function

$$|f(x+h_m)-f(x+h_n)| \leqq \epsilon_1$$

when $x=0$ and never lies in the range $\epsilon_1<\xi<\eta$; so it is always $\leqq \epsilon_1$. Hence $\lim_{k\to\infty} f(x+h_k)$ exists uniformly in x.

4. *Proof of Finiteness of P.* It will next be shown that P is finite. By the definition of P, we know that there are P numbers z_j^*, $(j=1, \cdots, P)$, for which it is possible to find sequences $h_1^{(j)}, h_2^{(j)}, \cdots$ of $\mathfrak{R}$ for which

$$\lim_{k\to\infty} f(h_k^{(j)}) = z_j^*; \tag{6}$$

moreover, the z_j^* are the only such numbers. If P is infinite, the set of numbers z_j^* must have a limit point, and we can choose Q and R such that the positive number $\phi=|z_Q^*-z_R^*|$ is less than or equal to half the number η which occurs in the implicit function theorem. We next choose a positive number δ which does not exceed the number ν occurring in the implicit function theorem and such that when $|w| \leqq \delta$,

$$|Z_x(w)-f(x)| \leqq \frac{\phi}{4}. \tag{7}$$

By the definition of $\mathfrak{R}$, we can choose an integer m so that

$$|f(h_m^{(Q)})-z_Q^*| \leqq \frac{\phi}{4} \quad \text{and} \quad |f(h_m^{(R)})-z_R^*| \leqq \frac{\phi}{4}, \tag{8}$$

and such that for all x and all z in $\mathfrak{S}$,

$$\begin{aligned} |F(x+h_m^{(Q)}, z)-F(x,z)| &\leqq \frac{\delta}{2}, \\ |F(x+h_m^{(R)}, z)-F(x,z)| &\leqq \frac{\delta}{2}. \end{aligned} \tag{9}$$

Since $F[h_m^{(Q)}, f(h_m^{(Q)})] = 0$ it follows from (9) that if

$$w_0 = F[h_m^{(R)}, f(h_m^{(Q)})], \tag{10}$$

then $|w_0| \leqq \delta \leqq \nu$.

Now $Z_{h_m^{(R)}}(w_0)$ is the one and only number which satisfies both $|Z_{h_m^{(R)}}(w_0) - f(h_m^{(R)})| \leqq \eta$, and $F[h_m^{(R)}, Z_{h_m^{(R)}}(w_0)] = w_0$. But $F[h_m^{(R)}, f(h_m^{(Q)})] = w_0$, and from (8) and the definition of ϕ, it can be shown that $|f(h_m^{(Q)}) - f(h_m^{(R)})| \leqq (3/2)\phi < \eta$. Thus $Z_{h_m^{(R)}}(w_0) = f(h_m^{(Q)})$; and, from (7), $|f(h_m^{(Q)}) - f(h_m^{(R)})| \leqq \phi/4$. Hence, from (8), we obtain $|z_Q^* - z_R^*| \leqq (3/4)\phi$, which contradicts the definition of ϕ and shows that P is finite.

5. *Uniformity of the Limits.* In this section it will be shown that for any sequence $h_1, h_2, \cdots$ of $\mathfrak{R}$, we have uniformly in x

$$\lim_{K \to \infty} f(x + Ph_K) = f(x). \tag{11}$$

As before, let $\lim_{K\to\infty} f(h_K^{(j)}) = z_j^*$, where $z_1^*, \cdots, z_P^*$ are the P numbers that can be obtained in this way. Then since $h_1, h_2, \cdots$ is in $\mathfrak{R}$, there exist functions $z_1(x), \cdots, z_P(x)$ such that uniformly in x, $\lim_{K\to\infty} f(x + h_K^{(j)}) = z_j(x)$, $(j = 1, \cdots, P)$, where of course $z_j(0) = z_j^*$, $(j = 1, \cdots, P)$. Now let $k_1^{(j)}, k_2^{(j)}, \cdots$ be an element of $\mathfrak{R}$ such that $\lim_{K\to\infty} f(k_K^{(j)}) = z_j^*$ and let $l_1^{(j)}, l_2^{(j)}, \cdots$ be the sequence $h_1^{(j)}, k_1^{(j)}, h_2^{(j)}, k_2^{(j)}, \cdots$. Clearly $l_1^{(j)}, l_2^{(j)}, \cdots$ is an element of $\mathfrak{R}$, and hence uniformly in x, $\lim_{K\to\infty} f(x + k_K^{(j)}) = z_j(x)$. Thus the P functions $z_1(x), \cdots, z_P(x)$ are the only functions $z(x)$ such that $\lim_{K\to\infty} f(x + k_K) = z(x)$ uniformly in x for some sequence $k_1, k_2, \cdots$ of $\mathfrak{R}$.

Because of the uniformity of the limits, we see that the sum and difference of any two elements of $\mathfrak{R}$ belongs to $\mathfrak{R}$; and that uniformly in x

$$\lim_{K\to\infty} z_i(x + h_K^{(j)}) = \lim_{K\to\infty} f(x + h_K^{(i)} + h_K^{(j)}), \quad (i = 1, \cdots, P).$$

But the right member is one of the $z_j(t)$, and hence translating by $h_K^{(j)}$ and taking the limit effects a permutation S_j on the P functions. Moreover, any other sequence $k_1, k_2, \cdots$ which takes $f(x)$ into $z_j(x)$ effects the same permutation S_j, and each sequence of $\mathfrak{R}$ effects one of the permutations $S_1, \cdots, S_P$. Finally, $h_K^{(i)} + h_K^{(j)}$ effects the permutation $S_i S_j = S_j S_i$; the sequence

0, 0, $\cdots$ effects the identity; and the sequence $-h_K^{(i)}$ effects the inverse of S_i; so these permutations form a group of order P. Hence the Pth power of each element is the identity, and (11) follows.

6. *Completion of the Proof.* The proof of the theorem will now be completed by considering any sequence $h_1, h_2, \cdots$ such that $\lim_{K\to\infty} F(x+h_K, z) = F(x, z)$ uniformly in x and z for all x and all z in $\mathfrak{C}$. Suppose that it is not true that uniformly in x

$$\lim_{K\to\infty} f(x + Ph_K) = f(x). \tag{12}$$

Then there exist $\epsilon>0$ and an infinite sub-sequence $h_1', h_2', \cdots$ of $h_1, h_2, \cdots$ and a sequence of numbers $x_1, x_2, \cdots$ such that for all K,

$$| f(x_K + Ph_K') - f(x_K) | > \epsilon. \tag{13}$$

Let $h_1'', h_2'', \cdots$ be a sub-sequence of $h_1', h_2', \cdots$ such that $f(h_1''), f(h_2''), \cdots$ converges. Then $h_1'', h_2'', \cdots$ is an element of $\mathfrak{R}$, and we have, uniformly in x, $\lim_{K\to\infty} f(x+Ph_K'') = f(x)$. But this contradicts (13) and hence (12) holds uniformly in x. It follows that, uniformly in x, $\lim_{K\to\infty} f[P(x+h_K)] = f(Px)$; and hence that $f(Px)$ is a.p. Moreover, the module of $f(Px)$ is contained in the module of $F(x, z)$, and the theorem follows.

7. *Algebraic Functions.* Let $D(a_1, \cdots, a_n)$ denote the discriminant of $z^n+a_1z^{n-1}+\cdots+a_n=0$; let $\phi_1(x), \cdots, \phi_n(x)$ be u.a.p. functions such that $|D[\phi_1(x), \cdots, \phi_n(x)]| > \delta>0$ for all x; and let $F(x, z) = z^n+\phi_1(x)z^{n-1}+\cdots+\phi_n(x)$. If $f(x)$ is any one of the n continuous solutions of

$$F[x, f(x)] = 0,$$

then it is clear that $f(x)$ and $F(x, z)$ satisfy the conditions of our theorem, except possibly the condition that for all x, $|F_z[x, f(x)]| > \lambda > 0$. That this condition also holds can be seen by considering the closures $\mathfrak{K}$ and $\mathfrak{L}$ of the n- and $(n+1)$-dimensional vectors $[\phi_1(x), \cdots, \phi_n(x)]$ and $[f(x), \phi_1(x), \cdots, \phi_n(x)]$. If $[b, a_1, \cdots, a_n]$ is in $\mathfrak{L}$,

$$b^n + a_1b^{n-1} + \cdots + a_n = 0,$$

and $|D[a_1, \cdots, a_n]| \geqq \delta > 0$, so that

$$nb^{n-1} + (n-1)a_1b^{n-2} + \cdots + a_{n-1} \neq 0.$$

Since $\mathfrak{P}$ is closed and bounded, our condition follows. We can now draw the conclusion of our theorem. In particular, since the limit of a sequence of $\mathfrak{R}$ is a solution of $F(0, z)=0$, we have $P \leqq n$. Thus we see that $f(x)$ is u.a.p. and its module is contained in the set of numbers obtained by dividing each element of the module of $\phi_1(x), \cdots, \phi_n(x)$ by P. This theorem was proved by Walther,* except that his characterization of the module is less incisive in that he uses $n!$ in place of $P \leqq n$.

8. *Generalization.* Theorem 1 may be generalized by allowing z to be an n-dimensional vector $z_1, \cdots, z_n$ and $f(x)$ and $F(x, z)$ vector functions $f_1(x), \cdots, f_n(x)$ and $F_1(x, z), \cdots, F_n(x, z)$. The f_i, F_i, and z_i are all real or all complex, while x remains a single real variable. The distance between z and z' is to be $\max_i |z_i - z_i'|$, and similarly for f and F. The derivative $\partial F/\partial z$ is to be replaced by all the partial derivatives $\partial F_i/\partial z_j$ in (c) and (e), and by

$$\frac{\partial[F_1, \cdots, F_n]}{\partial[z_1, \cdots, z_n]}$$

in (d). The condition (e) is to be strengthened by requiring uniform boundedness at the stipulated points as well as uniform continuity; and in view of this, the condition (b) may be omitted from the statement of the theorem. The original proof holds without change, and need not be repeated. If we omit the part of the theorem concerning the modules, we can also allow x to become a point in a connected space E as described by Bochner in his paper *Abstracte fastperiodische Funktionen.*†

9. *Stepanoff Functions.* In the following theorem n-dimensional vectors will be used, and the symbol $\|P\|$ will denote the norm $\Sigma|p_i|$ of the vector $P: p_1, \cdots, p_n$. Vectors will be added, multiplied by scalars, differentiated, and integrated by having the operation performed on each component.

THEOREM 2. *Let* $\Phi(z, P) \equiv \Phi(z, p_1, p_2, \cdots, p_n) \equiv z^n + p_1 z^{n-1} + \cdots + p_n$, *and let* $\Delta(P)$ *be the discriminant of* $\Phi(z, P)=0$. *Let* $F(t)$ *be an S.a.p. vector function such that*

* *Algebraische Funktionen der fastperiodischen Funktionen,* Monatshefte für Mathematik, vol. 40 (1933), pp. 444–457.

† Acta Mathematica, vol. 61, pp. 149–184, especially p. 150.

$$\phi(t, u) = \max_{0<v\leq u} \left\| F(t) - \frac{1}{v}\int_t^{t+v} F(\xi)d\xi \right\|$$

converges in measure as $u\to 0$ *uniformly on every interval of unit length; and let positive numbers* λ *and* δ_1 *exist such that*

$$\left| \Delta\left[\frac{1}{u}\int_t^{t+u} F(\xi)d\xi\right] \right| \geqq \lambda$$

for all positive $u \leqq \delta_1$ *and all* t. *Then the equation* $\Phi[z(t), F(t)] = 0$ *has an S.a.p. solution.*

It should be noted that the condition on $\phi(t, u)$ is a restriction only so far as the uniformity of the convergence is concerned. As $u\to 0$, $\phi(t, u)$ must automatically converge in measure on every bounded interval.

To establish this theorem, let I be the interval $-\infty < t < +\infty$, R_0 the two-dimensional set $-\infty < t < +\infty$, $0 < u \leqq \delta$, and R the set consisting of R_0 together with the points $t \subset I - E_0$, $u = 0$, where E_0 is the set of measure zero on which it is not true that

$$\lim_{u\to 0^+} \frac{1}{u}\int_t^{t+u} F(\xi)d\xi = \frac{d}{dt}\int_0^t F(\xi)d\xi = F(t).$$

Let

$$\frac{1}{u}\int_t^{t+u} F(\xi)d\xi = F(t, u),$$

and $F(t) = F(t, 0)$; then $F(t, u)$ is a continuous function of u on R and is continuous in t and u on R_0. Since R_0 is simply connected, it follows that if we choose a value for the solution of $\Phi[z(t, u), F(t, u)]$ at one point, say $t = 0$, $u = \delta$, and require that $z(t, u)$ be continuous, then $z(t, u)$ is uniquely defined on our strip; and if we define $z(t, 0)$ as $\lim_{u\to 0^+} z(t, u)$, the function $z(t, u)$ is uniquely defined on R. Moreover $|\Delta[F(t, u)]| \geqq \lambda$ on R; it follows that if we change the value of $z(0, \delta)$ to another root of $\Phi[z, F(0, \delta_1)] = 0$, the value of $z(t, u)$ will be changed at every point R. Thus the two values of $z(t, 0)$ will differ on a set of positive measure.

Suppose now that the value of $z(0, \delta)$ has been chosen and that $z(t, u)$ is accordingly defined on R. Let $z(t, 0)$ be defined

arbitrarily on E_0 so as to satisfy $\Phi[z, F(t)]=0$. Then $z(t, 0)$ is a solution of this equation for all t. It remains to prove that this solution is S.a.p.

Let $\epsilon>0$, and let L be an $(\epsilon/9)$-inclusion interval of $F(t)$. Let $\eta \leqq \epsilon/9$ be a positive number such that if S is any measurable subset of the interval $0 \leqq t \leqq L+1$ whose measure does not exceed η, then $\int_S \|F(\xi)\| d\xi \leqq \epsilon/9$. Let N be so great that

$$\sup_t m[\|F(\xi)\| > N]_t^{t+1} \leqq \frac{\eta}{2}.$$

Here $m[S]_a^b$ denotes the measure of the part of the set S included in the interval $a \leqq \xi \leqq b$. Such a choice of N is possible since $\int_t^{t+1} \|F(\xi)\| d\xi$ is bounded. Let I_t denote the interval $t \leqq \xi \leqq t+1$, and let E_1 be the set of values of t such that $\|F(t)\| > N$. Then on $I-(E_0+E_1)$, $|\Delta[F(t)]| \geqq \lambda$ and $F(t)$ is bounded; so by the implicit function theorem there exist positive numbers h and r, such that whenever $\|P-F(t)\| \leqq r$, $t \subset I-(E_0+E_1)$, then there exists one and only one complex number $Z_t(P)$ which satisfies simultaneously $\Phi[Z_t(P), P]=0$ and $|Z_t(P)-z(t, 0)| \leqq h$. Moreover, we can choose a positive number ν which does not exceed r or $\epsilon/9$ such that whenever $\|F(t)-P\| \leqq \nu$, $t \subset I-(E_0+E_1)$, then $|Z_t(P)-z(t, 0)| \leqq \epsilon/9$. Now choose $\bar{\delta}$ such that when $0 \leqq u \leqq \bar{\delta}$, then for all t, $m[\|\phi(\xi, u)\| > \nu]_t^{t+1} \leqq \eta/2$; and such that on this range, $0 \leqq u \leqq \bar{\delta}$, we have for all t

$$\int_t^{t+1} \|F(\xi) - F(\xi, u)\| d\xi \leqq \frac{\epsilon}{9}.$$

Finally, let E_2 be the set of values of t at which $\|\phi(t, \bar{\delta})\| > \nu$. Then $\sup_t m[I_t E_2] \leqq \eta/2$ and $\sup_t m[I_t(E_0+E_1+E_2)] \leqq \eta$.

Let t be any real number and let t equal $s+\tau$, where τ is an $(\epsilon/9)$-translation number of $F(t)$ and $0 \leqq s \leqq L$. If $f_1(\xi, u), \cdots, f_n(\xi, u)$ are the components of $F(\xi, u)$, then it follows from $0=\Phi[z(\xi, u), F(\xi, u)]$ that whenever $|z(\xi, u)| \geqq 1$, $|z(\xi, u)| = |-f_1(\xi, u)-f_2(\xi, u)[z(\xi, u)]^{-1}- \cdots -f_n(\xi, u)[z(\xi, u)]^{1-n}| \leqq \|F(\xi, u)\|$. Thus for all ξ, u, we have $|z(\xi, u)| < 1+\|F(\xi, u)\| \leqq 1+\|F(\xi-\tau, 0)\|+\|F(\xi, 0)-F(\xi-\tau, 0)\|+\|F(\xi, u)-F(\xi, 0)\|$, and for $0 \leqq u \leqq \bar{\delta}$,

$$\int_{I_t(E_0+E_1+E_2)} |z(\xi, u)|\, d\xi \leqq m[I_t(E_0+E_1+E_2)]$$
$$+ \int_{I_t(E_0+E_1+E_2)} \|F(\xi-\tau)\| d\xi + \int_t^{t+1} \|F(\xi)-F(\xi-\tau)\| d\xi$$
$$+ \int_t^{t+1} \|F(\xi, u)-F(\xi)\| d\xi \leqq \eta + \frac{3\epsilon}{9} \leqq \frac{4\epsilon}{9}$$

since

$$\int_{I_t(E_0+E_1+E_2)} \|F(\xi-\tau)\| d\xi = \int_S \|F(\xi)\| d\xi,$$

where $m(S) \leqq \eta$ and $S \subset I_{t-\tau}$ which is contained in the interval $0 \leqq \xi \leqq L+1$. Thus for $0 \leqq u \leqq \bar{\delta}$,

$$\int_{I_t(E_0+E_1+E_2)} |z(\xi, 0)-z(\xi, u)|\, d\xi \leqq \frac{8\epsilon}{9}. \tag{4}$$

Now on $I-(E_0+E_1+E_2)$, $\|\phi(\xi, \bar{\delta})\| \leqq \nu$; and whenever $\|P-F(\xi)\| \leqq \nu$, then $|Z_\xi(P)-z(\xi, 0)| \leqq \epsilon/9$. Therefore we may write $\Phi\{Z_\xi[F(\xi, u)], F(\xi, u)\}=0$, $\Phi[z(\xi, u), F(\xi, u)]=0$, and $|Z_\xi[F(\xi, u)]-z(\xi, 0)| \leqq \epsilon/9$ if $u \leqq \bar{\delta}$ and $\xi \subset I-(E_0+E_1+E_2)$. But for a fixed ξ in $I-(E_0+E_1+E_2)$, $z(\xi, u)-Z_\xi[F(\xi, u)]$ is continuous for $u \leqq \bar{\delta}$ and is zero when $u=0$. Thus by the uniqueness of the solution $Z_\xi(P)$ in its neighborhood, we have $z(\xi, u)=Z_\xi[F(\xi, u)]$, and hence $|z(\xi, u)-z(\xi, 0)| \leqq \epsilon/9$ when $0 \leqq u \leqq \bar{\delta}$ and $\xi \subset I-(E_0+E_1+E_2)$. Hence for $0 \leqq u \leqq \bar{\delta}$,

$$\int_{I_{t-(E_0+E_1+E_2)}} |z(\xi, 0)-z(\xi, u)|\, d\xi \leqq \int_{I_t} \frac{\epsilon}{9} d\xi \leqq \frac{\epsilon}{9},$$

and $\int_{I_t} |z(\xi, 0)-z(\xi, u)|\, d\xi \leqq \epsilon$. This inequality does not depend on t and we have that $z(\xi, 0)$ is the Stepanoff limit of $z(\xi, u)$, and hence is S.a.p.

COROLLARY. *Under the conditions of the theorem, the equation* $\Phi[z(t), F(t)]=0$ *will always have an infinite number of essentially distinct S.a.p. solutions, provided* $n>1$.

For it is clear from the argument used in the theorem that $\Phi[z(t), F(t)]=0$ has at least n essentially distinct solutions $z_1(t), \cdots, z_n(t)$, no two of which are equal on a set of positive

measure. Let L be a positive number and let $z(t, L)$ be equal either to $z_1(t)$ or $z_2(t)$ according as $[t/L]$ is even or odd. It is clear that $z(t, L)$ is an S.a.p. solution of $\Phi[z(t), F(t)] = 0$ for each positive L, and that all these solutions are essentially distinct. Of course many other solutions could be constructed in a similar way.

BROWN UNIVERSITY AND PRINCETON UNIVERSITY

ON A LEMMA OF FEJÉR*

BY LINCOLN LA PAZ

1. *Simple Integrals.* In an important paper L. Fejér† has verified and used the following lemma.

LEMMA A. *If for a problem of minimizing an integral*

$$I_1 = \int_{x_1}^{x_2} \phi(y') \cdot f(x, y) dx, \tag{1}$$

the Euler equation in normal form is

$$y'' = F(x, y, y'), \tag{1'}$$

then for a problem of minimizing the integral

$$J_1 = \int_{x_1}^{x_2} [\phi(y')/f(x, y)] dx, \tag{2}$$

the Euler equation in normal form is

$$y'' = -F(x, y, y'). \tag{2'}$$

The following generalization of Fejér's lemma is proved in this note.‡

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* Presented to the Society, December 2, 1933.

† L. Fejér, *Das Ostwaldsche Prinzip in der Mechanik*, Mathematische Annalen, vol. 61 (1905), p. 432.

‡ In everything that follows, the range of the indices i, j, k, μ, ν is from 1 to n and μ and ν are umbral.

LEMMA B. *If for a non-singular problem of minimizing an integral*

$$(3) \qquad I_n = \int_{x_1}^{x_2} \phi(y_1', \cdots, y_n') f(x, y_1, \cdots, y_n) dx,$$

with integrand function different from zero and of class C'', the system of Euler equations in normal form is*

$$(3') \qquad y_k'' = F_k(x, y_1, \cdots, y_n, y_1', \cdots, y_n'),$$

then for a problem of minimizing the integral

$$(4) \qquad J_n = \int_{x_1}^{x_2} \phi(y_1', \cdots, y_n') \Phi(f) dx,$$

where Φ is different from zero and of class C'', but is otherwise an arbitrary function of its argument, the system of Euler equations in normal form is

$$(4') \qquad y_k'' = (f\Phi'/\Phi) F_k(x, y_1, \cdots, y_n, y_1', \cdots, y_n').$$

In (4'), Φ' denotes the derivative of Φ with respect to its argument.

This lemma is an immediate consequence of the fact that the Euler systems of the problems (3) and (4) can be put into simple normal forms in terms of the transversality coefficients $t_i = (\partial\phi/\partial y_i')/(\phi - y_\mu' \partial\phi/\partial y_\mu')$ of these problems.† In fact, the Euler equations of (3) and (4) can be written $y_k'' = d_k/D$ and $y_k'' = D_k/D$, respectively. Here $D = |\partial t_i/\partial y_j'|$ and d_k and D_k are the determinants obtained from D by replacing the element $(\partial t_i/\partial y_k')$ by

$$[\partial(\log \Phi)/\partial y_i - t_i \partial(\log \Phi)/\partial x](1 + y_\mu' t_\mu)$$

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* The term class is used in the sense of Bolza, *Vorlesungen über Variationsrechnung*, pp. 13–14. As C. Carathéodory has recently pointed out (*Bermerkung über die Eulerschen Differentialgleichungen der Variationsrechnung*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1931), p. 40) the classical assumption that the integrand function is of class $C^{(n)}$, $n \geqq 3$, is unnecessary.

† L. La Paz, *The Euler equations of problems of the calculus of variations with prescribed transversality coefficients*, Proceedings of the National Academy of Sciences, vol. 17 (1931), p. 471, (17).

with $\Phi=f$ and $\Phi=\Phi(f)$, respectively. Since $D_k=f(\Phi'/\Phi)d_k$, (k), the lemma follows.*

2. *Application to Brachistochronic Integrals.* As an application of the lemma just established, consider the problem of obtaining a normal Euler system for the important class of brachistochronic integrals in $(x, y_1, \cdots, y_n)$-space

$$(5) \qquad T = \int_{s_1}^{s_2} \frac{ds}{v(r)} = \int_{x_1}^{x_2} \frac{(1 + y_\mu' y_\mu')^{1/2}}{v([x^2 + y_\mu y_\mu]^{1/2})}\, dx,$$

the extremals of which may be interpreted as the trajectories of light corpuscles moving through a medium in which the velocity of propagation, v, depends alone on the distance from a fixed center or, under suitable assumptions, as the paths of seismic rays in the interior of the earth. Examination of the values assigned to $v(r)$, for example by Kummer in his classical paper on astronomical refraction† or by Kövesligethy, Wiechert, and others in connection with the problem of the seismic brachistochrone,‡ will show that even in the case $n=1$ the direct determination of normal Euler equations for problems of the form (5) may be a matter of tedious calculation.

However, if we note that when $v(r)=r^{-1}$ the normal Euler system for (13) has the simple form

$$(6) \qquad y_i'' = r^{-2}(1 + y_\mu' y_\mu')(y_i - x y_i'), \quad (i = 1, 2, \cdots, n),$$

then the generalized lemma shows that the normal Euler system when $v=v(r)$ is simply

$$(7) \qquad y_i'' = -\,[v'(r)/rv(r)]B_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'),$$

where $B_i \equiv (1+y_\mu' y_\mu')(y_i - x y_i')$ and $i=1, 2, \cdots, n$.§

3. *Multiple Integrals.* For n-fold integrals of the form

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* The lemma may also be established by direct calculation of the Euler equations of (3) and (4) and subsequent solution for the derivatives y_i''.

† E. E. Kummer, *Über atmosphärische Strahlenbrechung*, Journal für Mathematik, vol. 61 (1863), p. 263.

‡ See, for example, V. Conrad, *Dynamische Geologie*, Encyklopädie der Mathematische Wissenschaften, vol. VI, 1, 11, 10, p. 453.

§ Compare L. La Paz, loc. cit., p. 461, (19).

$$\int_{(n)} f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1 \cdots dx_n, \tag{8}$$

we can state a result similar to that embodied in the generalized lemma of Fejér. The Euler-Lagrange partial differential equation for a problem of minimizing an integral (8) is

$$f_{p_\mu x_\mu} + f_{p_\mu z}p_\mu + f_{p_\mu p_\nu}p_{\mu\nu} - f_z = 0, \tag{9}$$

where $p_i = \partial z/\partial x_i$ and $p_{ij} = \partial^2 z/\partial x_i \partial x_j$. Consider now two integrals (8) with integrand functions of the special forms

$$\begin{aligned} f_1 &= a(x_1, \cdots, x_n, z)b(p_1, \cdots, p_n), \\ f_2 &= \Phi(a)b(p_1, \cdots, p_n), \end{aligned} \tag{10}$$

where a, b and Φ are different from zero and of class C'', but are otherwise arbitrary functions. We then have the following lemma.

LEMMA C. *If the Euler-Lagrange equation* (9) *of a regular problem of minimizing an integral* (8) *with integrand function* $f = f_1$ *is expressed in the form*

$$b_{p_\mu p_\nu}p_{\mu\nu} = F(x_1, \cdots, x_n, z, p_1, \cdots, p_n), \tag{11}$$

then the Euler-Lagrange equation of a similar problem with integrand function $f = f_2$ *is expressible in the form*

$$b_{p_\mu p_\nu}p_{\mu\nu} = (a\Phi'/\Phi)F(x_1, \cdots, x_n, z, p_1, \cdots, p_n). \tag{12}$$

This result can be justified by an argument based on considerations analogous to those employed in the first section or by a direct verification which can be made under even less restrictive hypotheses than those of the lemma.

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ON APPROXIMATION TO AN ANALYTIC FUNCTION BY POLYNOMIALS*

BY O. J. FARRELL

1. *Introduction.* If the function $f(z)$ is analytic in the interior R of a Jordan curve C and continuous in the corresponding closed region $\overline{R}$, then in that closed region $f(z)$ can be uniformly expanded in a series of polynomials in z. This result is due to Walsh,† who recently encouraged the writer to study the problem of polynomial approximation to a function $f(z)$ analytic interior to C but not necessarily continuous in $\overline{R}$ or even bounded in R. Professor Walsh suggested (1) that if $f(z)$ is bounded in R, there might exist polynomials $p_n(z)$, $(n=1, 2, \cdots)$, which approximate to $f(z)$ in R in such a way that the limit as $n\to\infty$ of the least upper bound of $|p_n(z)|$ for z in R does not exceed the least upper bound of $|f(z)|$ for z in R; (2) that if the double integral over R of $|f(z)|^p$, $(p>0)$, exists, there might be a sequence of polynomials $\{p_n(z)\}$ which approximate to $f(z)$ in R so that the limit as $n\to\infty$ of the double integral over R of $|f(z)-p_n(z)|^p$ is zero. It is the purpose of the present note to show that the existence of such approximating polynomials can be established in both cases by a method due to Carleman‡ not only for a Jordan region but also for regions of somewhat more general boundary.

It is a well known characteristic of a finite simply connected region R bounded by a Jordan curve C that the totality of points of the extended plane not belonging to $R+C$ form a single region (also simply connected) whose boundary is precisely C. But the class of regions characterised by this property includes not only all finite Jordan regions but regions of more general boundary as well.§ And it has been found that the results sug-

* Presented to the Society, March 31, 1934.

† Walsh, Mathematische Annalen, vol. 69 (1926), p. 430.

‡ Carleman, Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1923), pp. 1–30, §1.

§ For an example of such a region with a boundary which is not a Jordan curve see, for instance, Carathéodory, Mathematische Annalen, vol. 73, pp. 305–370, §35.

gested by Walsh hold for an arbitrary region of this class. More precisely stated the results obtained are given by the following two theorems.

THEOREM 1. *In the z-plane let G be a finite simply connected region whose boundary B is such that all the points of the extended plane not on G+B form a single region whose boundary is precisely B. If the function f(z) is analytic and bounded in G, then there exist polynomials* $p_n(z)$, $(n=1, 2, \cdots)$, *which converge to f(z) continuously* in G as* $n\to\infty$ *and for which*

$$(1)\quad \{\lim_{n\to\infty}\{\overline{\text{bound}}\,[\,|\,p_n(z)\,|, z \text{ in } G]\}\} \leqq \overline{\text{bound}}\,[\,|\,f(z)\,|, z \text{ in } G].$$

THEOREM 2. *In the z-plane let G be a finite simply connected region of the kind described in Theorem* 1. *If the function f(z) is analytic in G and such that*

$$\iint_G |\,f(z)\,|^p dS, \qquad (0 < p),$$

exists, then there exist polynomials $p_n(z)$, $(n=1, 2, \cdots)$, *which converge to f(z) continuously in G as* $n\to\infty$ *and such that*

$$(2)\quad \lim_{n\to\infty} \iint_G |\,f(z) - p_n(z)\,|^p dS = 0.$$

In a subsequent paper the writer hopes to tell what is the most general finite simply connected region to which each of these two theorems can be extended.

2. *Proof of Theorem* 1. Let G_n, $(n=1, 2, \cdots)$, be a set of uniformly bounded simply connected regions such that the closed region $\overline{G}=G+B$ lies in every G_n and the region G is the kernel† of every infinite subsequence of the sequence $\{G_n\}$. Let the function $\psi_n(z)$ map the region G_n conformally onto the region G in such a way that a fixed point z_0 in G and a fixed direction at z_0 remain invariant. It follows then from results of Bieberbach‡ that

* See Walsh, Transactions of this Society, vol. 33 (1931), pp. 668–689, p.669.

† See Carathéodory, Mathematische Annalen, vol. 72 (1912), pp. 107–144, p. 124, or Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 1927, pp. 12–13.

‡ Bieberbach, loc. cit., p. 13.

$$\lim_{n\to\infty} \psi_n(z) = z, \qquad \lim_{n\to\infty} \psi_n'(z) = 1$$

continuously in G. Hence the functions

$$f_n(z) = f[\psi_n(z)], \qquad (n = 1, 2, \cdots),$$

are analytic in the closed region $\overline{G}$ and they converge to $f(z)$ continuously in G. Moreover, it is to be noted that

$$\overline{\text{bound}}\ [\,|f_n(z)|, z \text{ in } G] \leqq \overline{\text{bound}}\ [\,|f(z)|, z \text{ in } G]$$

for every n, since the set of values which $f_n(z)$ takes on in G is precisely the set of values which $f(z)$ takes on in the transform of G by the mapping relation $\psi_n(z)$, and this transform of G lies interior to G.

By a theorem of Walsh* the function $f_n(z)$ can be uniformly approximated as closely as desired in $\overline{G}$ by a polynomial in z. If we choose polynomials $\{p_n(z)\}$ so that $|f_n(z) - p_n(z)| \leqq \epsilon_n$ for z in $\overline{G}$, where $\lim_{n\to\infty} \epsilon_n = 0$, we have at once a set of polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G and which satisfy (1).

We have not used here the fact that the derivative of $\psi_n(z)$ converges to unity continuously in G as $n\to\infty$. This fact will, however, be required in the proof of Theorem 2.

3. *Proof of Theorem* 2. We require for later application in the proof of this theorem the following lemma on double integrals.

LEMMA. *If the functions $f(z)$ and $\phi(z)$ are continuous in a region R of the z-plane and such that*

$$\iint_R |f(z)|^p dS = M, \qquad \iint_R |\phi(z)|^p dS = N, \quad (0 < p),$$

then

$$\iint_R |f(z) - \phi(z)|^p dS \leqq 2^p (M + N). \tag{3}$$

* Walsh, loc. cit., bottom of page 433 and top of page 434.

To prove this lemma consider the real continuous non-negative function $g(z)$ which is equal to $|f(z)|$ when $|f(z)| = |\phi(z)|$ and is equal to the greater of these two absolute values when they are unequal. It follows at once from the definition of $g(z)$ that the double integral over R of $[g(z)]^p$ exists and that

$$\iint_R [g(z)]^p dS \leqq M + N. \tag{4}$$

But since $|f(z)-\phi(z)| \leqq |f(z)| + |\phi(z)| \leqq 2g(z)$, it is clear that (3) follows from (4), and the lemma is proved.

Let $\{G_n\}$ and $\{\psi_n(z)\}$ denote respectively the regions used in the proof of Theorem 1 and their mapping functions. Let Γ_n denote the region into which G is transformed by $\psi_n(z)$. If the proper determination of each of the functions

$$F_n(z) = f[\psi_n(z)][\psi_n'(z)]^{2/p}, \qquad (n = 1, 2, \cdots),$$

be chosen (and we will suppose that this has been done), then the functions $F_n(z)$ are analytic everywhere in the closed region $\overline{G}$ and $\lim_{n\to\infty} F_n(z) = f(z)$ continuously in G. Moreover

$$\lim_{n\to\infty} \iint_G |F_n(z)|^p dS = \lim_{n\to\infty} \iint_{\Gamma_n} |f(z)|^p dS = \iint_G |f(z)|^p dS.$$

But by the theorem of Walsh used in the proof of Theorem 1 each function $F_n(z)$ can be uniformly approximated as closely as desired in $\overline{G}$ by a polynomial in z. So it is clear that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G and for which

$$\lim_{n\to\infty} \iint_G |p_n(z)|^p dS = \iint_G |f(z)|^p dS. \tag{5}$$

It remains to show that these polynomials also satisfy (2). This will be done by application of the lemma together with the fact that the convergence of the polynomials to $f(z)$ is continuous in G. Let ϵ be an arbitrarily small positive number. Let H be any subregion of G such that the closed region $\overline{H}$ lies interior to G and also such that

$$\iint_{G-H} |f(z)|^p dS < \epsilon. \tag{6}$$

The polynomials $\{p_n(z)\}$ established in the preceding paragraph are such that from a certain value of n on, say for $n_0 \leqq n$, the following three conditions are simultaneously satisfied:

$$\left| \iint_G |p_n(z)|^p dS - \iint_G |f(z)|^p dS \right| < \epsilon, \tag{7}$$

$$|p_n(z) - f(z)| < \epsilon, \qquad (z \text{ in } \overline{H}), \tag{8}$$

$$\left| \iint_H |p_n(z)|^p dS - \iint_H |f(z)|^p dS \right| < \epsilon. \tag{9}$$

From (6), (7), and (9) it follows that

$$\iint_{G-H} |p_n(z)|^p dS < 3\epsilon, \qquad (n_0 \leqq n).$$

Let $A = \iint_G dS$. Then

$$\begin{aligned} \iint_G |f(z) - p_n(z)|^p dS &= \iint_H |f(z) - p_n(z)|^p dS \\ &\quad + \iint_{G-H} |f(z) - p_n(z)|^p dS \\ &\leqq A\epsilon^p + 2^p(4\epsilon), \end{aligned}$$

where the second integral on the right has been appraised by the lemma. The quantity $m(\epsilon) = A\epsilon^p + 2^p(4\epsilon)$ is clearly a function of ϵ which approaches zero as limit along with ϵ. Consequently the polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G and for which (5) holds must also satisfy (2).

4. *A Comparison of the Two Theorems.* It may not be without interest to compare these two theorems with each other and with Runge's* fundamental theorem on polynomial approximation, at least in the case of a simple example. For our example we take $p = 1$, $f(z) = z$, and for G the interior of the unit circle, $|z| < 1$. The polynomials $p_n(z) = nz^n + z$, $(n = 1, 2, \cdots)$, converge to $f(z)$ continuously in G, but they do not satisfy (2), for

* See, for instance, Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, 1928, p. 578.

$$\lim_{n\to\infty} \iint_G |f(z) - p_n(z)|\, dS = \lim_{n\to\infty} \int_0^{2\pi} \int_0^1 nr^{n+1} dr d\theta, \qquad (z = re^{i\theta}),$$

$$= \lim_{n\to\infty} \frac{2\pi n}{n+2} = 2\pi.$$

On the the other hand the polynomials $P_n(z) = z^n + z$ not only converge to $f(z)$ continuously in G but also satisfy (2), as may be readily verified. They do not, however, satisfy (1), for $\overline{\text{bound}}\,[|P_n(z)|, z \text{ in } G] = 2$ for every n. Finally the polynomials $\pi_n(z) = (z^n/n) + z$ converge to $f(z)$ continuously in G and at the same time satisfy (1).

5. *A Remark on Theorem* 1. *Theorem* 1 *can not be extended to the most general finite simply connected region.* Essentially the same remark was made without amplification by Carleman in the paper cited. The writer feels, however, that the following detailed justification of the remark is not unworthy of consideration. A region such that Theorem 1 can not hold for an arbitrary function $f(z)$ analytic and bounded therein is any finite simply connected region R such that the complete boundary of R is not the boundary of an infinite region, as for instance a circle from which one radius has been removed or again a region consisting of the points of a strip which lies inside a circle, is closed at one end, and winds around infinitely often, approaching as limit every point of the circle. Let B denote that portion of the boundary of R which is the boundary of an infinite region. All the points of the plane which can be joined to a fixed point of R by a Jordan arc containing no point of B form a region S which contains not only all the points of R but boundary points of R as well. Let $f(z)$ be a function which maps R conformally onto the interior of the unit circle. If polynomials $\{p_n(z)\}$ existed which converged to $f(z)$ in R and for which (1) held, it would follow from the continuity of these polynomials in the closed region $\overline{R}$ that the $\{p_n(z)\}$ would be uniformly bounded in S and hence form a normal family in S.* A subsequence of these polynomials could then be found which would

* See Montel, *Leçons sur les Familles Normales de Fonctions Analytiques*, 1927, pp. 21–33.

converge to a function $\phi(z)$ analytic in S and such that $|\phi(z)|<1$ for z in S. Since the original sequence $\{p_n(z)\}$ converges by hypothesis in R, the limit function of the subsequence would necessarily agree with $f(z)$ in R. But $\phi(z)$ could not be identical with $f(z)$ in R, for then $\phi(z)$ would have to equal unity at interior points of S, namely, at the boundary points of R which lie interior to S.

It would appear from this remark that Theorem 1 (and Theorem 2 as well) might admit of extension to an arbitrary finite simply connected region whose complete boundary is also the boundary of an infinite region, and that a modification of the methods used in the proofs of the present paper would serve to establish such an extension. The writer hopes to answer this question in a later paper.

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NOTE ON A SIMPLE TYPE OF ALGEBRA IN WHICH THE CANCELLATION LAW OF ADDITION DOES NOT HOLD

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1. *Introduction.* I do not imagine that the algebraic system considered in this note can be new, but if it has been overlooked this has probably happened because of its simplicity. However, we shall be most interested here in examining the connection of the system with the foundations of ordinary algebra. As we shall see, the symbols employed have most of the properties of rational integers, the principal exceptions being that they are finite in number and from

$$a+b=a+c$$

we cannot infer $b=c$ in general.*

2. *Description of the System.* Suppose we introduce the natural numbers 1, 2, 3, $\cdots$, employing for their use Peano's system

* In a system in which we may always infer $b=c$ under the condition given we shall say the cancellation law holds.

of axioms. Instead of building up ordinary arithmetic by considering the operations of addition and multiplication on the natural numbers, we may introduce symbols

$$C_1, C_2, C_3, \cdots, \tag{1}$$

a relation symbolized by $(+)$ and another by $(=)$, and such that

$$C_1 + C_1 = C_2, \quad C_2 + C_1 = C_3, \quad \cdots, \quad C_n + C_1 = C_{n'}, \quad \cdots,$$

where n denotes a natural number and n' its immediate successor in the set of natural numbers. We also write

$$C_m + C_{n'} = (C_m + C_n) + C_1,$$

and by employing equality axioms and induction we obtain from this the associative and commutative laws of addition, involving the C's in a similar manner to that, for example, employed by Pringsheim.*

We introduce an operation, multiplication, such that

$$C_m C_{n'} = C_m C_n + C_m, \qquad C_m C_1 = C_m,$$

and employing these we may prove by induction, as did Pringsheim, the distributive law, the associative law, and the commutative law of multiplication involving the C's.

All the above mentioned arguments are independent of any possible equalities among the C's. If we assume that $C_m = C_n$ if and only if m and n denote the same natural numbers, then consider the relations

$$C_i + C_j = C_k \qquad \text{and} \qquad C_i C_j = C_m.$$

Suppose we replace any such relations by

$$i + j = k \qquad \text{and} \qquad ij = m;$$

then we have an example of isomorphism, there being a one-to-one correspondence between the subscripts of the C's and the natural numbers themselves under addition and multiplication.

This gives the elementary arithmetic of the natural numbers. We shall employ, for convenience, this arithmetic in what follows, although the system considered is really independent of it.

* *Vorlesungen über Zahlen und Funktionenlehre*, vol. 1, part 1, pp. 4–30.

Let us assume now that $C_1, C_2, \cdots, C_j$ are distinct, but $C_i = C_{j+1}$ with $j+1 > i$; then in the set (1) there are just j distinct elements

$$C_1, C_2, C_3, \cdots, C_i, \cdots, C_j, \tag{2}$$

since

$$C_{j+1} = C_i \quad \text{and} \quad C_{j+2} = C_{i+1},$$

and $i+1 \leqq j+1$. To indicate a certain property of the set (2) we shall consider a special case. Suppose that $C_1, C_2, \cdots, C_6$ are distinct and that $C_7 = C_4$; *then from* $C_6 + C_1 = C_3 + C_1$ *we cannot infer* $C_6 = C_3$; *that is, the cancellation law of addition does not hold in this algebra.*

We also see from the way in which it was introduced that *this system* (2) *is just as elementary and natural as ordinary arithmetic.* It is obviously not a ring in general. If we take the particular case, however, where $i = 1$ in (2), then

$$C_j + C_m = C_m$$

for $m = 1, 2, \cdots, j$, and also

$$C_j C_k = C_j,$$

so that C_j has the properties of a zero element, and the set (2) is isomorphic in this case to a complete system of residue classes, modulo j, and forms a ring.

3. *Semi-Groups.* To derive properties of this finite arithmetic (as well as rings) and to classify it among the different types of abstract algebras it is very convenient to employ the concept of *semi-group*, which will be defined as a set of elements which is closed under a certain operation symbolized by $\circ$, such that the associative law holds between the elements of the set combined under this operation; in other words, the set forms a group without the restriction that for A and B in the set there exists an X such that $A \circ X = B$, and a Y such that $Y \circ A = B$. Also the usual equality axioms hold.

If in addition we may infer from

$$A \circ B = A \circ C \quad \text{and} \quad B \circ D = C \circ D,$$

the relation

$$B = C,$$

A, B, C and D denoting any elements of the set, then we shall call the set a *quasi-group.** We shall now write, in any semi-group, AB for $A \circ B$, and A^n for $A \circ A \circ \cdots$ to n terms.

If an element I exists in the semi-group S such that $AI = A$ and $IA = A$ for any A in S, then I is called the *identity* or *unity* element of S. It is easily seen that there cannot be more than one identity element of a semi-group. A *unit* element of S (containing an identity element) is an element U in S such that there exists a U' and U'' in S for which $UU' = I$ and $U''U = I$.

An element C of any semi-group S is called *cancellable* if from $CA = CB$ and $MC = NC$, we may infer $A = B$ and $M = N$ for any A, B, M and N in S; otherwise we shall call it *non-cancellable.*

We shall now prove the following theorem.

THEOREM 1. *In order that a finite semi-group contain a cancellable element it is necessary and sufficient that it contain an identity element.*

Let

$$A_1, A_2, \cdots, A_k \tag{3}$$

be the distinct elements of a finite semi-group S and assume that C is a cancellable element of S; then

$$CA_1, CA_2, \cdots, CA_k \tag{3a}$$

is a permutation of (3), because $CA_i = CA_j$ gives $A_i = A_j$, since C is cancellable. Hence for a given A_n in (3) there is an A_x in (3) such that

$$CA_x = A_k. \tag{4}$$

If C is cancellable, so is C^r. In the set

$$C, C^2, C^3, \cdots$$

we must have repetitions since the set is finite. Let

$$C^t = C^{t+e},$$

with $e > 0$. If $t > 1$, then, since C^{t-1} is cancellable, we obtain

* Dickson, Transactions of this Society, vol. 6 (1905), calls this system a semi-group.

$$C = C^{e+1} \tag{5}$$

so that this relation holds for all cases. Now for a given h consider A_x from (4); we shall have from (5)

$$CA_x = C^e(CA_x),$$

and

$$A_h = C^e A_h. \tag{6}$$

Similarly, considering in lieu of (3a) the set

$$A_1C, A_2C, \cdots, A_kC,$$

we obtain

$$A_h = A_hC^e, \tag{7}$$

so that (6) and (7) show that $C^e = I$ is an identity element of S. The condition of the theorem is sufficient since from $AI = AI$ we have $A = A$ and similarly for $IA = IA$.

The relation (4) and its analogon $A_yC = A_h$ give the known result that any finite quasi-group is a group; hence my introduction of the term *quasi*.

4. *Properties of the System.* Consider a finite semi-group given by powers of a single element A. It may have the distinct elements

$$A, A^2, \cdots, A^i, \cdots, A^j,$$

with

$$A^{j+1} = A^i.$$

This set is isomorphic with the additive semi-group (2). The elements

$$A^i, \cdots, A^j$$

are known to form a group. (This is conveniently proved by first noting that they form a quasi-group.) Applying this statement to the finite algebra given by (2), we obtain the result that $C_i, C_{i+1}, \cdots, C_j$ form under addition and multiplication a ring.

Although there may be no zero elements in the set (2), yet

we may have non-cancellable elements in the semi-group formed by multiplication of the elements. For if $j=6$, $i=4$, we have

$$C_2C_2 = C_5C_2,$$

but we cannot infer

$$C_2 = C_5.$$

Hence, the system differs from a ring in this respect, for in the latter a non-cancellable element under multiplication is necessarily a zero-divisor.

If, in the relation (2), $i>1$, then there is no zero element in the set, or considered as an additive semi-group there is no identity element in the set; hence, by Theorem 1 there is no cancellable element under addition. This is in sharp contrast to the case when $i=1$, in which case we have a group under addition and all the elements are cancellable.

Considering the multiplicative semi-group of (2) for $i=1$, we have the case where the system is isomorphic with the residue classes, modulo j, so that the cancellable elements are known. For $i>1$ there is no cancellable element except C_1, for the cancellable elements form a quasi-group and therefore a group which includes C_1; hence, if C_a is cancellable, there is a C_b such that

$$C_aC_b = C_1,$$

which is impossible, since $ab>1$ and $i>1$.

The various properties we have established for (2) may be stated as follows.

THEOREM 2. *Under addition and multiplication with the assumptions given, the set* (2) *forms an algebra in which for* $i>1$, *the additive semi-group of the algebra contains no cancellable elements and the multiplicative semi-group contains no cancellable elements other than* C_1. *The elements* C_i, C_{i+1}, $\cdots$, C_j *form a ring which for* $i=1$ *is isomorphic with the system of all residue classes, modulo* j.

5. *Extensions and Modifications of the System.* We may obtain other algebras which have the property that cancellation under addition is not possible in general and multiplication is non-commutative by taking the system of all square matrices of order n whose elements are contained in (2).

Suppose we consider the natural numbers and consider a relation* between them which we shall call equivalence, symbolized by ($\cong$), such that if a and b are natural numbers and if $a<i$, then from $a\cong b$ it follows that $a=b$, and if a and $b\geqq i$, then $a\equiv b \pmod{m}$. For $m=j-i+1$, we obtain a system which is isomorphic with the C's in (2) provided equivalence takes the place of equality, the number of non-equivalent integers being j. Using this relation between rational integers, we may consider conditional equivalences, such as

$$x^2 \cong a,$$

where a is a positive integer. The question naturally arises as to possible generalizations of the law of quadratic reciprocity, since equivalence is an extension of the idea of congruence.

6. *The System as a Type of Algebra.* As to the place of our system among known types of associative algebras in which the usual distributive laws hold, we may first define a ring as a system of elements which form an Abelian group under an operation called addition, a semi-group under multiplication, and in which the right and left hand distributive laws hold. But number theorists have worked since the beginning of the history of their subject in systems which are not always rings, for example, in connection with the problem of expressing a positive integer as the sum of positive cubes. The positive integers under addition form an Abelian quasi-group in lieu of an Abelian group. *The next natural generalization is to replace quasi-group by semi-group, and if this is done, our system* (2) *exemplifies an algebra of the type where the elements form a semi-group under addition, but not a quasi-group.*

We may, indeed, proceed further and define a *semi-ring* or *associative algebra* as a set of elements forming a semi-group under addition, a semi-group under multiplication, and in which the right and left hand distributive laws hold. This leads us naturally to the consideration of associative algebras in which addition is not commutative, that is, the restriction "Abelian" is removed in connection with the original definitions of addition. Certain of these types I hope to take up in another paper.

THE UNIVERSITY OF TEXAS

* As suggested by A. Church.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

PITTSBURGH, PENNSYLVANIA, Annual Meeting, December 27-31, 1934.

This meeting is to be held in conjunction with Section A of the A.A.A.S. and the M.A.A. Professor A. B. Coble will deliver his retiring presidential address. There will be a joint session with the American Physical Society, at which time there will be a symposium on mathematical physics with special reference to quantum mechanics. Professor Albert Einstein will deliver the eleventh Josiah Willard Gibbs Lecture. At a joint session with Section A of the A.A.A.S. and the Mathematical Association, addresses will be delivered by Professor C. N. Moore as retiring Chairman of Section A, on *Mathematics and Science*, and by Professor Arnold Dresden as retiring President of the Mathematical Association, on *A Program for Mathematics*.

NEW YORK CITY, February 23, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than January 26. Abstracts received by the Secretaries on or before January 11 will appear in the January issue of this Bulletin. By invitation of the Program Committee, Professor Morgan Ward will deliver an address on the topic *Arithmetic Theory of linear recurring series*.

STANFORD UNIVERSITY, April 6, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than March 9. All these will appear in the March issue of this Bulletin.

NEW YORK CITY, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. There will be a symposium on the topic *The problem of three bodies, and allied problems*, at which the speakers will be Professors G. D. Birkhoff, E. W. Brown, and Aurel Wintner.

CHICAGO, ILLINOIS, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. By invitation of the Program Committee, Professor L. M. Graves will deliver in address on *Topics in Functional Calculus*.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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For official Communications and Notices, see the inside of the back cover.

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